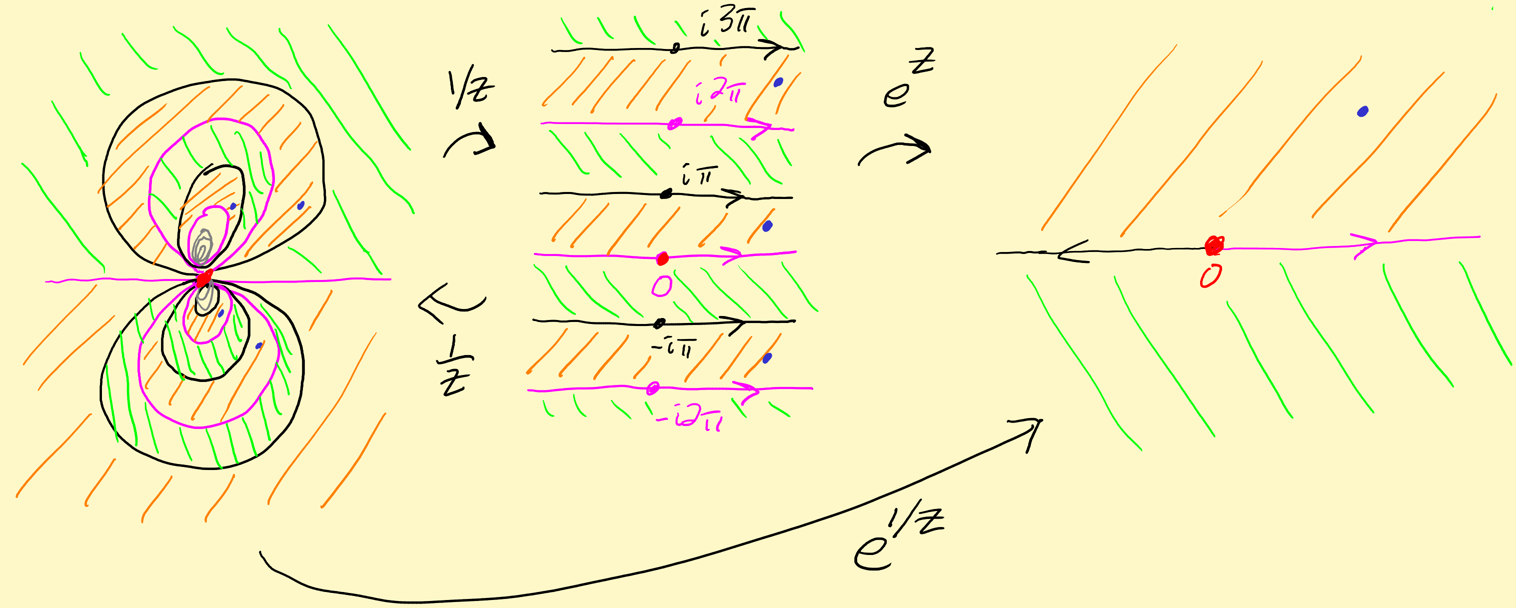
Lesson 27 Isolated singularities



Wow! $e^{1/z}$ maps $D_\varepsilon(0)-\{0\}$ onto $\mathbb{C}-\{0\}$ infinite-to-one!

Def^n: z_0 is called an isolated singularity of an analytic fcn $f(z)$ if there is an $\varepsilon>0$ such that f is analytic on $D_\varepsilon(z_0)-\{z_0\}$.

Three types of singularities z_0 is an isolated sing of f.

Know $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converging on $D_\varepsilon(z_0)-\{z_0\}$.

3 cases: 1) $a_n=0$ for all negative n.

z_0 is a removable singularity:

Why: $f(z)$ = Laurent expansion = Taylor series $\longrightarrow$ analytic fcn on $D_\varepsilon(z_0)$. Aha! Set $f(z_0)=a_0$ to see that f can be given a value at z_0 to make it analytic on $D_\varepsilon(z_0)$.

EX: $\frac{\sin z}{z}$

z_0 removable for f $\iff$ $\lim_{z \to z_0} f(z)$ exists

$\iff$ f is bounded on

$D_\rho(z_0) - \{z_0\}$ for some $0 < \rho < \varepsilon$.

Case 2: Only finitely many $a_n \neq 0$ with negative n.

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{z-z_0} + \underbrace{a_0 + a_1(z-z_0) + \cdots}_{\substack{\text{power series} \\ H(z)}}$$

z_0 is a pole of $f(z)$.

Hmmm: $$(z-z_0)^N f(z) = \underbrace{\underbrace{a_{-N} + a_{-N+1}(z-z_0) + \text{power series terms}}_{\substack{\text{converges for same } z \text{ as the} \\ \text{series for } H(z). \text{ R of C same!}}}}_{= F(z)}$$

Notice that $F(z_0) = a_{-N} \neq 0$, and

$$f(z) = (z-z_0)^{-N} F(z)$$ similar to zeroes!

See that $\lim_{z \to z_0} f(z) = \infty$ The defining property of a pole.

Def^n a_{-N} is the "first" non-zero negative coeff in Laurent expansion $(N \geq 1)$. N = "order of the pole of f at z_0"

EX: $\dfrac{\sin z}{z^4} = \dfrac{1}{z^3} - \dfrac{1/3!}{z} + \left(\dfrac{1}{5!}\right)z - \cdots$

Case 3 Infinitely many negative terms $\neq 0$.

z_0 is an essential singularity of f.

EX: $e^{1/z}$

Picard's (big) theorem: Suppose f has an essential sing at z_0, f is analytic on $D_r(z_0) - \{z_0\}$. For any $\varepsilon > 0$ with $0 < \varepsilon < r$, f maps $D_\varepsilon(z_0) - \{z_0\}$ infinite-to-one onto $\mathbb{C}$ with one possible exception [like $e^{1/z}$ misses $z=0$].

EX: What kind of singularity does $\cos \frac{1}{z}$ have at $z=0$?

Hmmm: For t real, $\cos \frac{1}{t}$ wiggles like crazy as $t \to 0$.

Conclude: $\lim\limits_{z \to 0} \cos \frac{1}{z}$ can't exist. So $z=0$ is not removable.

$\lim_{z \to 0} \cos \frac{1}{z}$ can't $= \infty$ either!

So $z=0$ is <u>not</u> a pole.

$z=0$ must be an <u>essential</u> sing.

<u>Riemann removable singularity theorem</u>: Suppose $f(z)$ is <u>analytic</u> and <u>bounded</u> on $D_r(z_0) - \{z_0\}$.

Then z_0 is a <u>removable</u> singularity.

[f bndd means $\exists\, M > 0$ such that $|f(z)| < M$ for all $z \in D_r(z_0) - \{z_0\}$.]

<u>Pf</u>: f <u>bndd</u> means z_0 is <u>not</u> a pole.

<u>and</u> z_0 is <u>not</u> essential either because it can't plaster the plane being bounded by M, violating Picard's. Only possibillity left: z_0 is removable.

<u>Def^n</u> Suppose f has an isolated sing at z_0, analytic on $D_r(z_0) - \{z_0\}$. The <u>residue of f at z_0</u> is a_{-1} from the Laurent expansion for f at z_0.

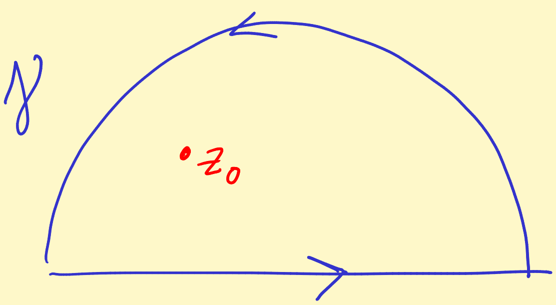
$$a_n = \frac{1}{2\pi i}\int_{C_\rho(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (0<\rho<r)$$

Notation: $$\operatorname{Res}_{z_0} f = a_{-1} = \frac{1}{2\pi i}\int_{C_\rho(z_0)} f(z)\,dz$$

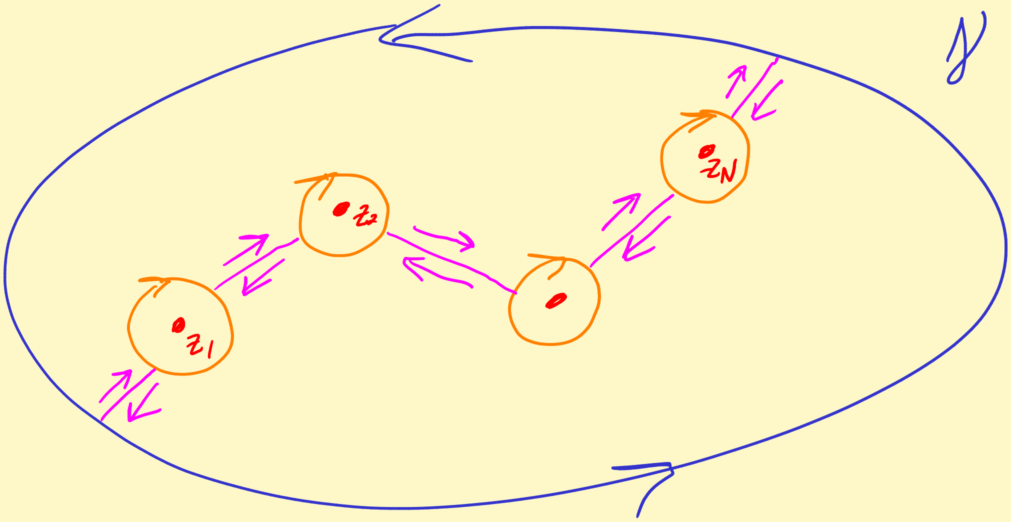
Baby residue theorem: $$\int_{C_\rho(z_0)} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0} f$$

Importance of the residue

Suppose
$$f(z) = \underset{\frac{d}{dz}\left[\frac{A_N}{-N+1}(z-z_0)^{-N+1}\right]}{\frac{A_N}{(z-z_0)^N}} + \frac{A_{N-1}}{(z-z_0)^{N-1}} + \cdots + \underset{\frac{d}{dz}\left[\frac{-A_2}{(z-z_0)}\right]}{\frac{A_2}{(z-z_0)^2}} + \overset{\operatorname{Res}_{z_0} f}{\frac{A_1}{z-z_0}} + \underset{\text{entire}}{g(z)}$$



$$\int_\gamma f\,dz = \underset{\text{Fund thm calc}}{0 + 0 + \cdots + 0} + \underbrace{2\pi i A_1}_{\text{Cauchy integral using } h(z)\equiv A_1} + \underset{\text{Cauchy thm}}{0}$$



Aha! $\quad 0 \underset{\substack{\uparrow \\ \text{Cauchy} \\ \text{thm}}}{=} \left(\int_{\text{Top}} + \int_{\text{Bottom}} \right) f \, dz$

$$= \int_\gamma f \, dz + \sum_{n=1}^{N} \underbrace{\int_{-C_\rho(z_n)} f \, dz}_{-2\pi i \, \mathrm{Res}_{z_n} f}$$

Baby residue thm.