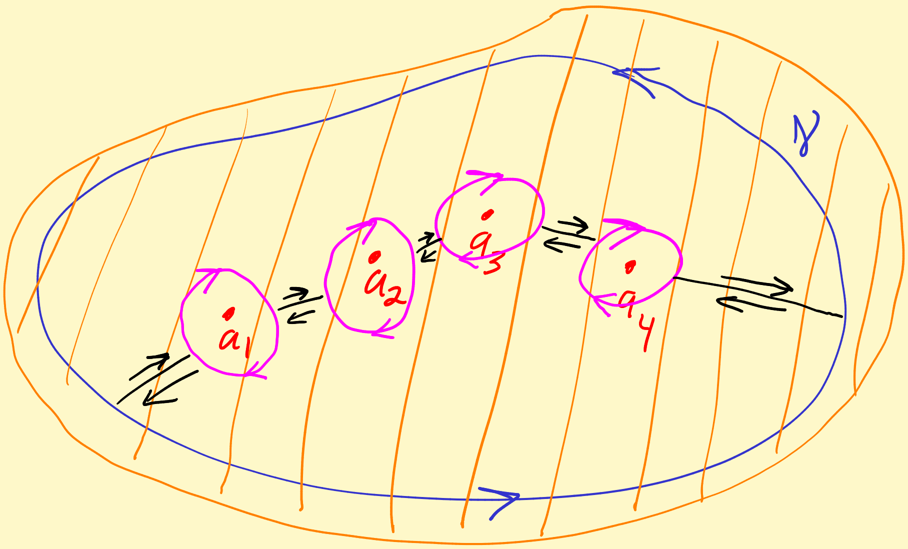
Lesson 28 The residue theorem



f analytic inside and on a simple closed curve γ, except at finitely many isolated singularities inside γ $\{a_n\}_{n=1}^{N}$

Then $$\int_\gamma f\,dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{a_n} f$$

Pf: $$0 = \Bigg(\underbrace{\int_{\text{Top half}} f\,dz}_{=0} + \underbrace{\int_{\text{Bottom half}} f\,dz}_{=0} \Bigg) =$$

Cauchy's thm

$$= \int_\gamma f\,dz + \sum_{n=1}^{N} \underbrace{\int_{-C_\varepsilon(a_n)} f\,dz}_{= -\int_{C_\varepsilon(a_n)} f\,dz = -2\pi i \operatorname{Res}_{a_n} f}$$

Applications of residue theorem

Famous integrals: 1) Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

Biggy: $\widehat{f'}(s) = -is\,\hat{f}(s)$

$\widehat{f''}(s) = -s^2\,\hat{f}(s)$

Fourier transf : $\{\text{Diff eqns}\} \longrightarrow \underbrace{\{\text{algebra}\}}_{\text{solve for } \hat{f}}$

Undo $\hat{f}$ to get f. Mind blowing fact:

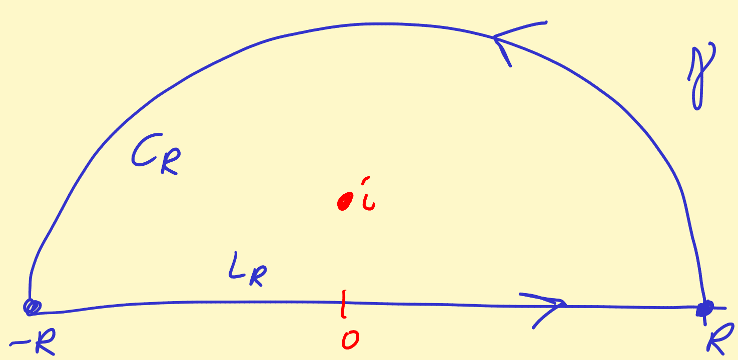
$$f(x) = c\int_{-\infty}^{\infty} e^{isx}\,\hat{f}(s)\,ds$$

Fourier inversion formula

EX: Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\,dx \qquad (s>0)$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{\cos sx}{x^2+1}}_{\text{even}}\,dx + \underbrace{i\int_{-\infty}^{\infty} \underbrace{\frac{\sin sx}{x^2+1}}_{\text{odd}}\,dx}_{=0}$$

$$= 2\int_{0}^{\infty} \frac{\cos sx}{x^2+1}\,dx$$



$$f(z) = \frac{e^{isz}}{z^2+1}$$

Res thm: $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \, \mathrm{Res}_i \frac{e^{isz}}{z^2+1}$

L_R : $z(t) = t$, $-R \le t \le R$. $z'(t) = 1$

$$\int_{L_R} f\,dz = \int_{-R}^{R} \frac{e^{ist}}{t^2+1}\, 1 \cdot dt \quad \longleftarrow \text{want!}$$

Claim: $\displaystyle\left| \int_{C_R} f(z)\,dz \right| \longrightarrow 0$ as $R \to \infty$.

Note $\left| e^{is(x+iy)} \right| = \left| e^{isx} e^{-sy} \right| = \underbrace{\left| e^{isx} \right|}_{=1} \cdot e^{-sy}$

$= e^{-sy} \le 1$ when $y \ge 0$ $(s > 0)$.

So $\left| e^{isz} \right| \le 1$ on C_R (in upper half plane).

and $|z^2+1| \ge \underbrace{\big|\, |z^2| - 1 \,\big|}_{\underbrace{|R^2-1|}_{=R^2-1 \text{ when } R>1.}}$

denom est

$|z| = R$ on C_R.
Will let $R \to \infty$.
Assume $R > 1$ now.

Basic estimate for integrals: $\displaystyle\left| \int_{C_R} \frac{e^{isz}}{z^2+1}\,dz \right| \le \left(\max_{C_R} \left| \frac{e^{isz}}{z^2+1} \right| \right) \mathrm{Length}(C_R)$

$$\le \frac{1}{R^2-1} \cdot \pi R$$

$\longrightarrow 0$ as $R \to \infty$.

Fantastic thing:

$$\int_{\gamma} = \int_{L_R} + \int_{C_R} = 2\pi i \, \mathrm{Res}_i \frac{e^{isz}}{z^2+1}$$

Let $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\,dx + 0 = 2\pi i \, \mathrm{Res}_i$$

How to compute Res: $\dfrac{e^{isz}}{z^2+1} = \dfrac{e^{isz}}{(z-i)(z+i)}$

$$= \frac{1}{z-i}\left[\underbrace{\frac{e^{isz}}{z+i}}_{\text{analytic in a disc about } i}\right]$$

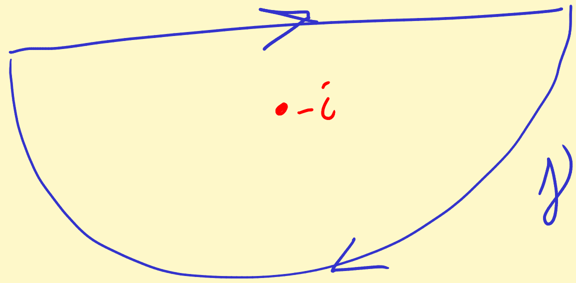
$A_0 + A_1(z-i) + A_2(z-i)^2 + \cdots = H(z)$

$$= \frac{A_0}{z-i} + A_1 + A_2(z-i) + \cdots$$

$$A_0 = \mathrm{Res}_i \frac{e^{isz}}{z^2+1} = H(i) = \frac{e^{isi}}{i+i} = \frac{e^{-s}}{2i}$$

Get $$\int_{-\infty}^{\infty} \frac{\cos sx}{x^2+1}\,dx = 2\pi i \frac{e^{-s}}{2i} = \pi e^{-s} \quad \text{when } s>0.$$

what if $s<0$? $\quad |e^{isz}| \le 1$ in lower half plane.



$$\int_\gamma f\,dz = -2\pi i \operatorname{Res}_{-i} f(z)$$

clockwise!

Important formula Suppose f and g are analytic on $D_r(a)$ and g has a simple zero at a (meaning that g has a zero of order $N=1$ at a: $g(a)=0$, but $g'(a)\neq 0$.)

Then $$\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$$

why

$$g(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

$a_0 = g(a) = 0 \qquad a_1 = \frac{g'(a)}{1!} \neq 0$

$$= (z-a)\underbrace{\left[a_1 + a_2(z-a) + \cdots\right]}_{G(z)}$$

$G(a) = a_1 = \frac{g'(a)}{1!} \neq 0$

$$\frac{f(z)}{g(z)} = \frac{1}{z-a}\cdot\underbrace{\frac{f(z)}{G(z)}}_{\text{analytic near } a}$$

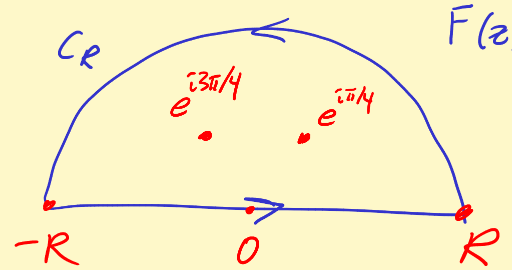
$$= \frac{1}{z-a}\Big[\underbrace{A_0 + A_1(z-a) + \cdots}_{H(z) \text{ analytic near } a}\Big]$$

$$= \frac{\overset{\text{Res}\,\frac{f}{g}}{A_0}}{z-a} + A_1 + A_2(z-a) + \cdots$$

$$\text{Res}_a \frac{f}{g} = A_0 = H(a) = \frac{f(a)}{G(a)} = \frac{f(a)}{\left[\frac{g'(a)}{1!}\right]} = \frac{f(a)}{g'(a)} \checkmark$$

EX: Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

$F(z) = \frac{1}{z^4+1}$



$$\left|\int_{C_R} \frac{1}{z^4+1}\,dz\right| \le \left(\max_{C_R}\left|\frac{1}{z^4+1}\right|\right)\cdot \pi R \qquad (R>1)$$

$$\le \frac{1}{R^4-1}\cdot \pi R \to 0 \text{ as } R\to\infty.$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left(\text{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \text{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1}\right)$$

$$F(z) = \frac{1}{z^4+1} \quad \begin{matrix} \leftarrow f(z) \equiv 1 \\ \leftarrow g(z) = z^4+1 \end{matrix}$$

$g'(z) = 4z^3$

$g(e^{i\pi/4}) = 0$

$g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 \neq 0$

Get $$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left[\frac{1}{4(e^{i\pi/4})^3} + \frac{1}{4(e^{i3\pi/4})^3}\right]$$

$$= \frac{2\pi i}{4}\left[e^{-i3\pi/4} + e^{-i9\pi/4}\right]$$

$$= \frac{\pi i}{2}\left[\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$$