Lesson 29 Applications of the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left[\frac{1}{4(e^{i\pi/4})^3}+\frac{1}{4(e^{i3\pi/4})^3}\right]$$

$$= \frac{2\pi i}{4}\left[e^{-i3\pi/4}+e^{-i9\pi/4}\right]$$

$$= \frac{\pi i}{2}\left[\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$$

Book

$$\frac{1}{z^4+1} = \frac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$= \frac{1}{z-e^{i\pi/4}}\underbrace{\left[\frac{1}{(\quad)(\quad)(\quad)}\right]}_{H(z)=A_0+A_1(z-e^{i\pi/4})+\cdots}$$

$$= \frac{\overset{\text{Res}_{e^{i\pi/4}}}{\boxed{A_0}}}{z-e^{i\pi/4}} + \text{power series}$$

$$\text{Res}_{e^{i\pi/4}}\,\frac{1}{z^4+1} = A_0 = H(e^{i\pi/4})$$

Ugh!

Important: Use $\text{Res}_a \dfrac{f(z)}{g(z)} = \dfrac{f(a)}{g'(a)} \leftarrow$ g has simple zero at a

<u>Theorem</u> P(x) and Q(x) polynomials,

$$\deg Q \geq \deg P + 2,$$

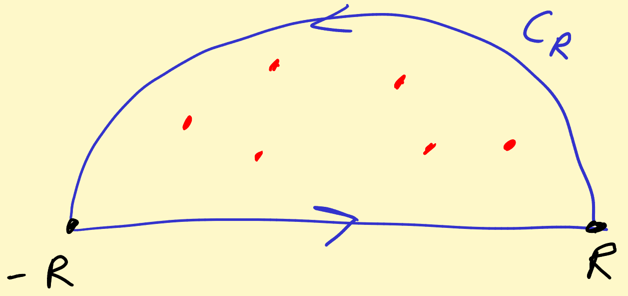
and Q(x) has no real zeroes.

Then $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a\in\mathfrak{A}} \text{Res}_a \frac{P(z)}{Q(z)}$$

(s > 0) $$\int_{-\infty}^{\infty} e^{isx}\frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a\in\mathfrak{A}} \text{Res}_a \left[e^{isz}\frac{P(z)}{Q(z)}\right]$$

where $\mathfrak{A} = \{a : a \text{ zeroes of } Q(z) \text{ in } \underbrace{\text{UHP}}_{\text{Upper Half Plane}}\}$

<u>Pf</u>:



Recall $|e^{isz}| \leq 1$ in UHP.

<u>Key</u>: Basic polynomial estimate:

$$a\,|z|^{\deg P} \le \boxed{|P(z)| \le A\,|z|^{\deg P}}\,, \quad |z| > R_P$$

$$\boxed{b\,|z|^{\deg Q} \le |Q(z)|} \le B\,|z|^{\deg Q}\,, \quad |z| > R_Q$$

Key: $$\left|\int_{C_R} \frac{P(z)}{Q(z)}\,dz\right| \le \left(\max_{C_R}\left|\frac{P(z)}{Q(z)}\right|\right)\left(\overset{\sim}{\pi} R\right)$$

$$\le \underbrace{\frac{A\,R^{\deg P}}{b\,R^{\deg Q}}}_{(A/b)\,\frac{1}{R^{\underbrace{\deg Q - \deg P}_{\ge 2}}}} \qquad \text{if } R > \mathrm{Max}(R_P, R_Q)$$

$$\le (A/b)\,\frac{1}{R^2} \qquad \text{if } R > \mathrm{Max}(1, R_P, R_Q)$$

$$\longrightarrow 0 \quad \text{as } R \to \infty$$

Same holds when e^{isz} is tossed in $(s>0)$.

<u>Remark</u>: $s<0$: $$\int_{-\infty}^{\infty} e^{isx}\,\frac{P(x)}{Q(x)}\,dx = -2\pi i \sum_{a\in\mathfrak{A}^-} \mathrm{Res}_a\, e^{isz}\,\frac{P(z)}{Q(z)}$$

$\mathfrak{A}^-$ = zeroes of Q in <u>Lower</u> half plane.

Trig integrals

EX: $I = \int_0^{\underline{\underline{2\pi}}} \cos^6 t \, dt$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{e^{it} + \frac{1}{e^{it}}}{2}$$

Hmmm. $C: \; z(t) = e^{it}, \quad 0 \le t \le 2\pi, \quad z'(t) = ie^{it}$

$$I = \int_0^{2\pi} \left[\frac{z(t) + \frac{1}{z(t)}}{2}\right]^6 \underbrace{\frac{1}{z'(t)} \boxed{z'(t) \, dt}^{\,dz}}_{\text{wish for } z'(t) \text{ here}}$$

$$= \int_0^{2\pi} \left[\frac{z(t) + \frac{1}{z(t)}}{2}\right]^6 \frac{1}{iz(t)} \underbrace{z'(t) \, dt}_{dz}$$

$$= \int_C \left[\frac{z + \frac{1}{z}}{2}\right]^6 \frac{1}{iz} \, dz$$

$$= 2\pi i \operatorname{Res}_0 \frac{1}{iz} \left[\frac{z + z^{-1}}{2}\right]^6$$

$$\frac{1}{iz} \cdot \left(\frac{z + z^{-1}}{2}\right)^6 = \frac{1}{2^6}\left(z^6 + 6z^4 + 15z^2 + \underline{\underline{20}} + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\right) \cdot \frac{1}{iz}$$

$$= \text{positive powers} + \frac{\boxed{20/(i2^6)}^{\,\text{Res}}}{z} + \text{negative powers, } -n, \; n > 1$$

$$I = 2\pi i \left(\frac{20}{i 2^6} \right) = \frac{5\pi}{8}$$

General method

$$I = \int_0^{\underline{\underline{2\pi}}} R(\cos\theta, \sin\theta)\, d\theta$$

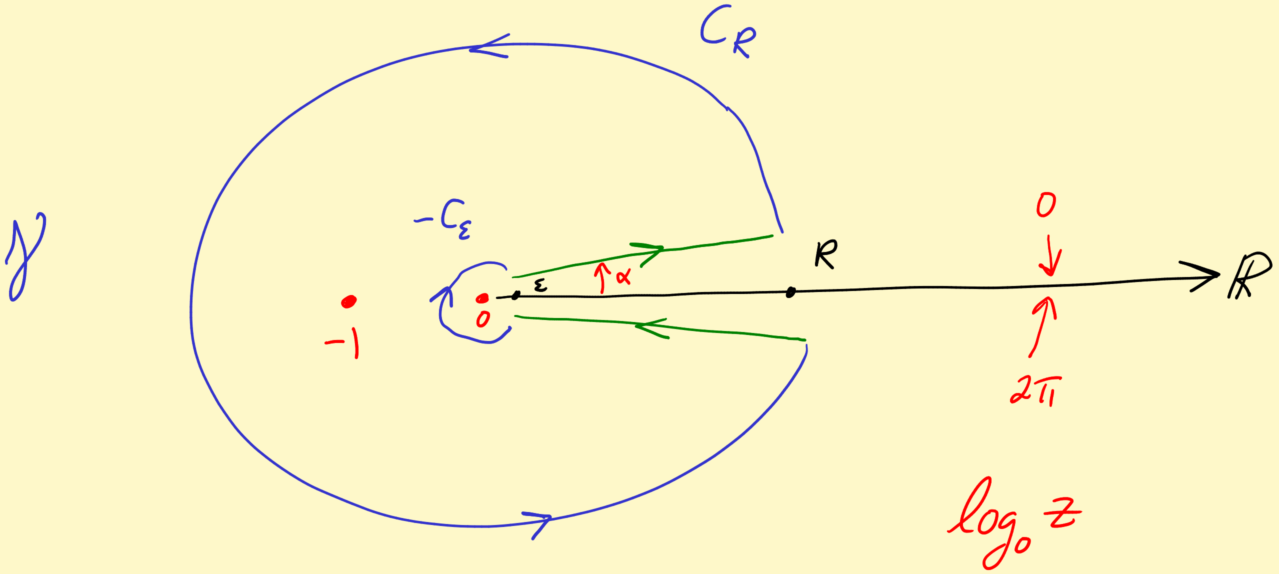
R rational without singularities on unit circle.

$$I = \int_C R\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i} \right) \cdot \frac{1}{iz} \, dz$$

$$= 2\pi i \sum \left(\text{Residues of integrand inside } C \right)$$

Integrals involving complex log fcns

$$I = \int_0^\infty \frac{1}{\sqrt{x}(x+1)} \, dx$$



$$z^{1/2} = e^{\frac{1}{2}\log_0 z}$$ where $\log_0 z = \mathrm{Ln}|z| + i\theta$

where $\theta \in \arg z$

with $0 < \theta < 2\pi$.

Plan $$f(z) = \frac{1}{z^{1/2}(z+1)}$$

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{-1} f(z)$$

$$f(z) = \frac{1/z^{1/2}}{z+1}$$ ← analytic near $z=-1$ ← simple zero at $z=-1$

$$\operatorname{Res}_{-1} f(z) = \frac{1/(-1)^{1/2}}{1} = \frac{1}{(-1)^{1/2}} = \frac{1}{e^{\frac{1}{2}(\mathrm{Ln}|-1| + i\pi)}}$$

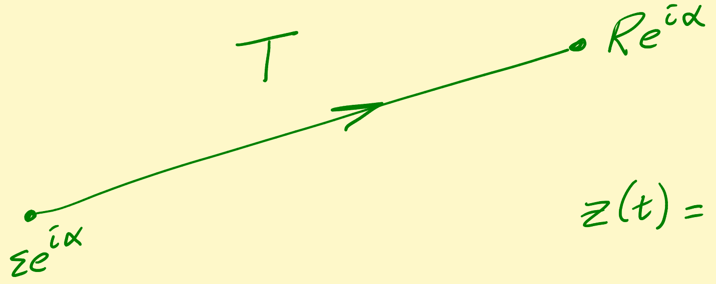
$$= \frac{1}{e^{i\pi/2}} = \frac{1}{i}$$

Plan: Let $\alpha \downarrow 0$. Let $\varepsilon \searrow 0$. Let $R \to \infty$.

1) $\displaystyle\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

2) $\displaystyle\int_{-C_\varepsilon} f(z)\,dz \to 0$ as $\varepsilon \searrow 0$

3)



$z(t) = te^{i\alpha}$, $\varepsilon \le t \le R$

$$\int_T f(z)\,dz = \int_\varepsilon^R f(te^{i\alpha})\, e^{i\alpha}\, dt$$

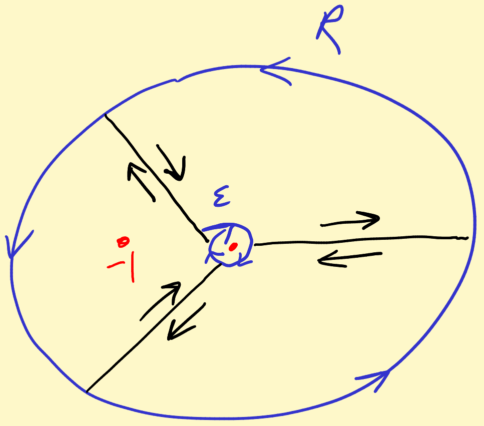
$$\xrightarrow[\alpha \downarrow 0]{} \int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\, dt$$

$$\int_{-B} f(z)\,dz = \int_\varepsilon^R f(te^{-i\alpha})\, e^{-i\alpha}\, dt$$

$$= \int_\varepsilon^R \frac{1}{\left[e^{\frac{1}{2}(\ln t + i2\pi)}\right](t+1)}\, dt$$

$$= e^{-i\pi} \int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\, dt$$

Book



Different branches of $z^{1/2}$ in pieces of pie