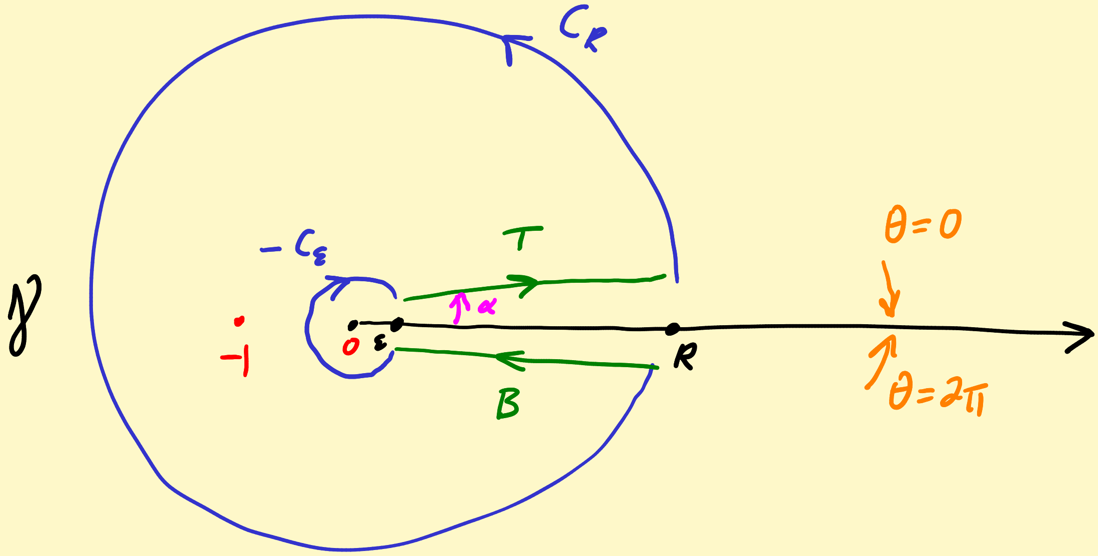
Lesson 30 More applications of the Residue theorem

$$I=\int_0^\infty \frac{1}{\sqrt{t}\,(t+1)}\,dt$$



$$z^{1/2}=e^{\frac{1}{2}\log_0 z}=e^{\frac{1}{2}(\mathrm{Ln}|z|+i\theta)},\quad \theta\in\arg z,\quad 0<\theta<2\pi$$

$$\int_\gamma \frac{1}{z^{1/2}(z+1)}\,dz=2\pi i\,\mathrm{Res}_{-1}\frac{1/z^{1/2}}{z+1}=2\pi i\,\frac{\frac{1}{(-1)^{1/2}}}{1}=\frac{2\pi i}{e^{\frac{1}{2}(\mathrm{Ln}|-1|+i\pi)}}$$

$$=\frac{2\pi i}{\underbrace{e^{i\pi/2}}_{=i}}=2\pi$$

Step 1 Let $\alpha\downarrow 0$

$$T:\ z(t)=te^{i\alpha},\quad \varepsilon\le t\le R\qquad z'(t)=e^{i\alpha}$$

$$\int_T f(z)\,dz=\int_\varepsilon^R \frac{1}{e^{\frac{1}{2}(\mathrm{Ln}\,t+i\alpha)}(te^{i\alpha}+1)}\,e^{i\alpha}\,dt$$

$\mathrm{Ln}\,t=\mathrm{Ln}|te^{i\alpha}|$

$$\xrightarrow[\alpha\downarrow 0]{}\int_\varepsilon^R \frac{1}{e^{\frac{1}{2}\mathrm{Ln}\,t}(t+1)}\,dt\qquad e^{\frac{1}{2}\mathrm{Ln}\,t}=t^{1/2}\ \checkmark$$

$-B: \; z(t) = t e^{-i\alpha}, \; \varepsilon \le t \le R \qquad z'(t) = e^{-i\alpha}$

$$\int_B = -\int_{-B} = -\int_\varepsilon^R \frac{1}{e^{\frac{1}{2}(\mathrm{Ln}\, t + i(2\pi - \alpha))}\left(t e^{-i\alpha} + 1\right)} e^{-i\alpha}\, dt$$

$$\xrightarrow[\alpha \searrow 0]{} -\int_\varepsilon^R \frac{1}{t^{1/2} \underbrace{e^{i\pi}}_{e^{i\pi} = -1} (t+1)}\, dt$$

Good news! $\int_T, \int_{-B}$ don't cancel! $(\alpha \searrow 0)$

Fact: $|z^{1/2}| = |z|^{1/2}$

Why: $\left| e^{\frac{1}{2}(\mathrm{Ln}|z| + i\theta)} \right| = e^{\frac{1}{2}\mathrm{Ln}|z|} \underbrace{\left| e^{i\theta/2} \right|}_{=1} = |z|^{1/2}$ ✓

Why #2: $w = z^{1/2}$

$w \cdot w = z$

$\underbrace{|w \cdot w|}_{|w|^2} = |z|$ So $|z^{1/2}| = |w| = |z|^{1/2}$.

Claim: $$\left| \int_{C_R} f(z)\, dz \right| \le \left(\max_{z \in C_R} \left| \frac{1}{z^{1/2}(z+1)} \right| \right) \underbrace{\mathrm{Length}(C_R)}_{< 2\pi R}$$

$|z| = R$: $\quad |z^{1/2}| = R^{1/2}$ ✓

$$|z+1| \geq \underbrace{\big|\underbrace{|z|}_{R} - 1\big|}_{R-1} \quad \longleftarrow \text{take } R>1$$

So $\displaystyle \left|\int_{C_R}\right| \leq \frac{1}{R^{1/2}(R-1)} \cdot 2\pi R \longrightarrow 0$ as $R \to \infty$.

<u>Claim</u> $\displaystyle \left|\int_{-C_\varepsilon}\right| \longrightarrow 0$ too as $\varepsilon \searrow 0$.

Same calculation, except

$|z| = \varepsilon$: $\quad \displaystyle |z+1| \geq \underbrace{\big|\underbrace{|z|}_{\varepsilon} - 1\big|}_{1-\varepsilon} \quad \longleftarrow \text{take } \varepsilon<1$

$$\left|\int_{-C_\varepsilon}\right| \leq \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \cdot 2\pi\varepsilon \longrightarrow 0 \text{ as } \varepsilon \searrow 0.$$

$$\int_{C_R} + \int_{-C_\varepsilon} + \underbrace{\int_T + \int_B} = 2\pi$$

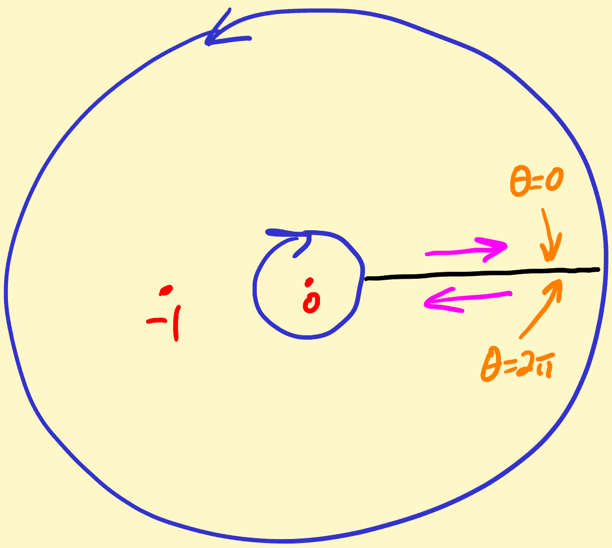
$$\alpha \to 0: \quad \int_{C_R} + \int_{-C_\varepsilon} + 2\int_\varepsilon^R \frac{1}{t^{1/2}(t+1)}\,dt = 2\pi$$

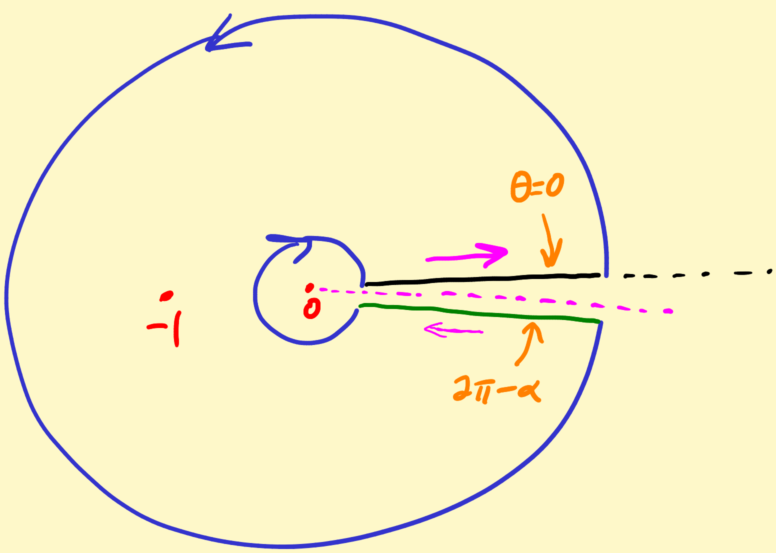
Let $R \to \infty$, $\varepsilon \searrow 0$

$$0 + 0 + 2I = 2\pi$$

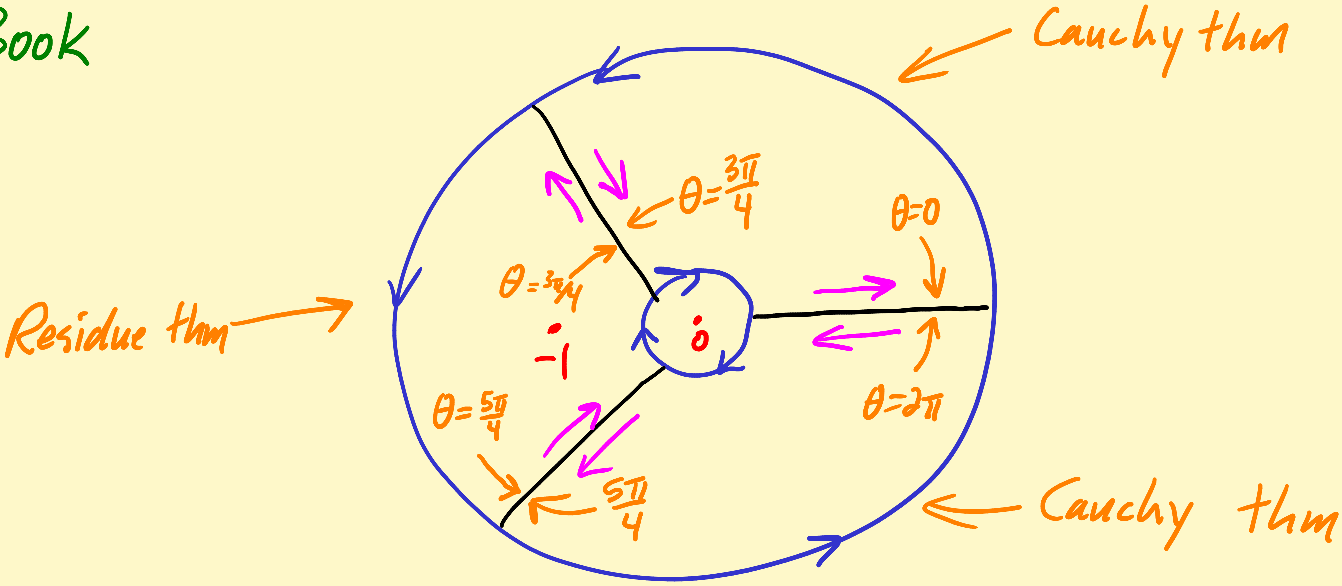
so $I = \pi$.

Other ways to think about same trick

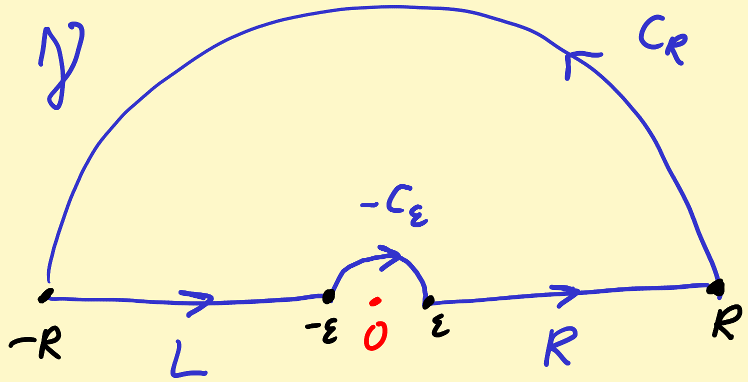




Book



Famous integrals $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$



Try

$f(z) = \dfrac{\sin z}{z}$ No!

Hmmm. $e^{it} = \cos t + i \sin t$

$$\frac{e^{it}}{t} = \underbrace{\frac{\cos t}{t}}_{\text{odd}} + i\,\underbrace{\frac{\sin t}{t}}_{\text{even}}$$

$$\boxed{f(z) = \frac{e^{iz}}{z}}$$

$$\left(\int_L + \int_R\right) f(z)\,dz = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right)\left(\frac{\cos t}{t} + i\,\frac{\sin t}{t}\right) 1 \cdot dt$$

$z(t) = t$
$z'(t) = 1$

$$= 0 + i\,2\int_{\varepsilon}^{R} \frac{\sin t}{t}\,dt$$ ← want

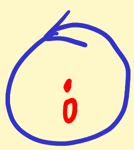
Cauchy's thm! $\displaystyle\int_{\gamma} \frac{e^{iz}}{z}\,dz = 0$

It works because:

1) $\displaystyle\int_{C_R} \frac{e^{iz}}{z}\,dz \to 0$ as $R \to \infty$.

2) $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz \to -i\pi$ as $\varepsilon \downarrow 0$.

How to remember (2):

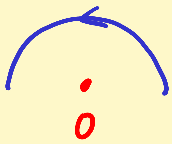
$-C_\varepsilon$ 

Residue thm: $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = -2\pi i \operatorname{Res}_0 \frac{e^{iz}}{z}$

clockwise

$$= -2\pi i \frac{e^{i\cdot 0}}{1} = -2\pi i$$

$-C_\varepsilon$



$\displaystyle\int_{-C_\varepsilon} =$ top half $\displaystyle\int_{-C_\varepsilon} \to \frac{1}{2}(-2\pi i) = -i\pi$ ✓

C_ε : $z(t) = \varepsilon e^{it}$, $0 \le t \le \pi$ $\qquad z'(t) = i\varepsilon e^{it}$

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = -\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz$$

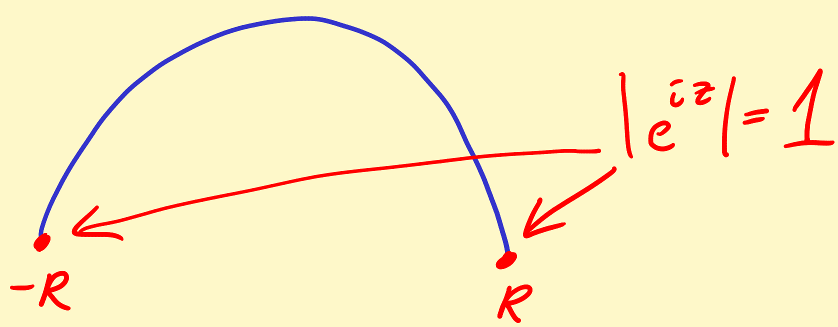
$$= -\int_0^{\pi} \frac{e^{i(\varepsilon\cos t + i\varepsilon\sin t)}}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt$$

$$= -i\int_0^{\pi} \underbrace{e^{-\varepsilon\sin t}\, e^{i\varepsilon\cos t}}_{\to 1 \text{ unif as } \varepsilon\to 0}\,dt$$

$$\longrightarrow -i\int_0^{\pi} 1\,dt = -i\pi$$

Hmmm. $$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \le \max_{C_R}\left|\frac{e^{iz}}{R}\right|\cdot \pi R$$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}e^{-y}| = e^{-y}$$



Ouch! Basic estimate fails us. $\left|\int_{C_R}\right| \le \pi\cdot 1$ Need Jordan's Lemma