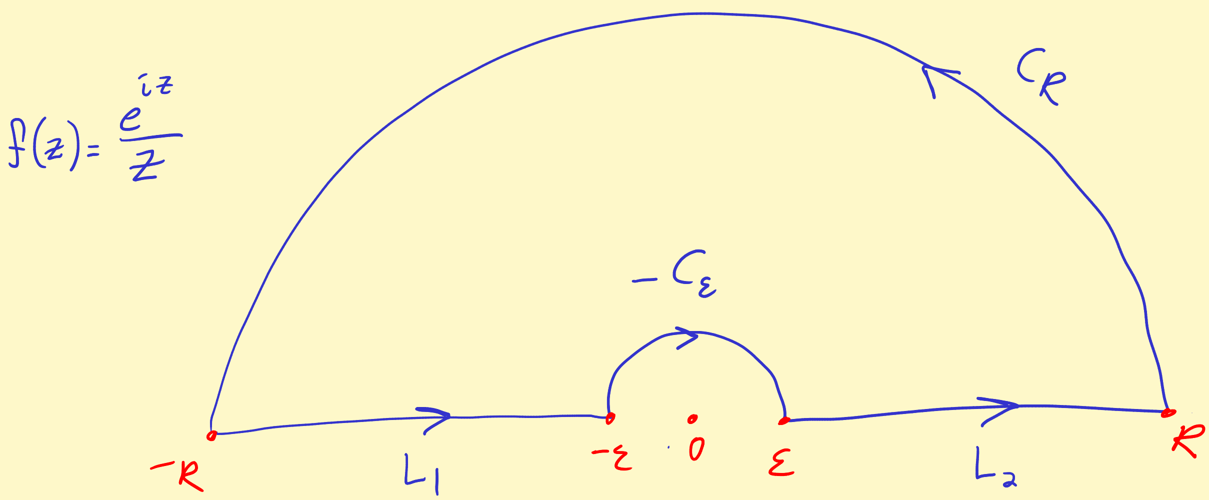
Lesson 31 Jordan's lemma



$$\left(\int_{L_1}+\int_{L_2}\right)\frac{e^{iz}}{z}\,dz = 2i\int_\varepsilon^R \frac{\sin t}{t}\,dt$$

Last time: $$\int_{-C_\varepsilon}\frac{e^{iz}}{z}\,dz \xrightarrow[\varepsilon\to 0]{} \frac{1}{2}\left[-2\pi i\,\underbrace{\operatorname{Res}_0 \frac{e^{iz}}{z}}_{1}\right]$$

halfway around Circle

clockwise

Easy way: $$\frac{e^{iz}}{z} = \frac{1+(iz)+\frac{(iz)^2}{2!}+\cdots}{z}$$

$$= \frac{1}{z} + \underbrace{i - \frac{z}{2!} + \cdots}_{\text{power series}}$$

$H(z)$ entire fcn.

$|H(z)| \le M$ on $\overline{D_1(0)}$.

$$\int_{-C_\varepsilon}\frac{e^{iz}}{z}\,dz = \int_{-C_\varepsilon}\frac{1}{z}\,dz + \int_{-C_\varepsilon}H(z)\,dz$$

$$-\int_0^{\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt = -i\pi$$

$$\left| \int_{-C_\varepsilon} H\, dz \right| \le \left(\underset{D_1(0)}{\mathrm{Max}}\, |H| \right) \pi\varepsilon \le M\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

$\varepsilon < 1$

Last thing we need: $\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| \to 0$ as $R \to \infty$.

(Jordan's lemma) Basic estimate fails us!

$$\int_{C_R} = \int_0^{\pi} \frac{e^{i(Re^{i\theta})}}{Re^{i\theta}}\, iRe^{i\theta}\, d\theta$$

$$= i\int_0^{\pi} e^{i[R\cos\theta + iR\sin\theta]}\, d\theta$$

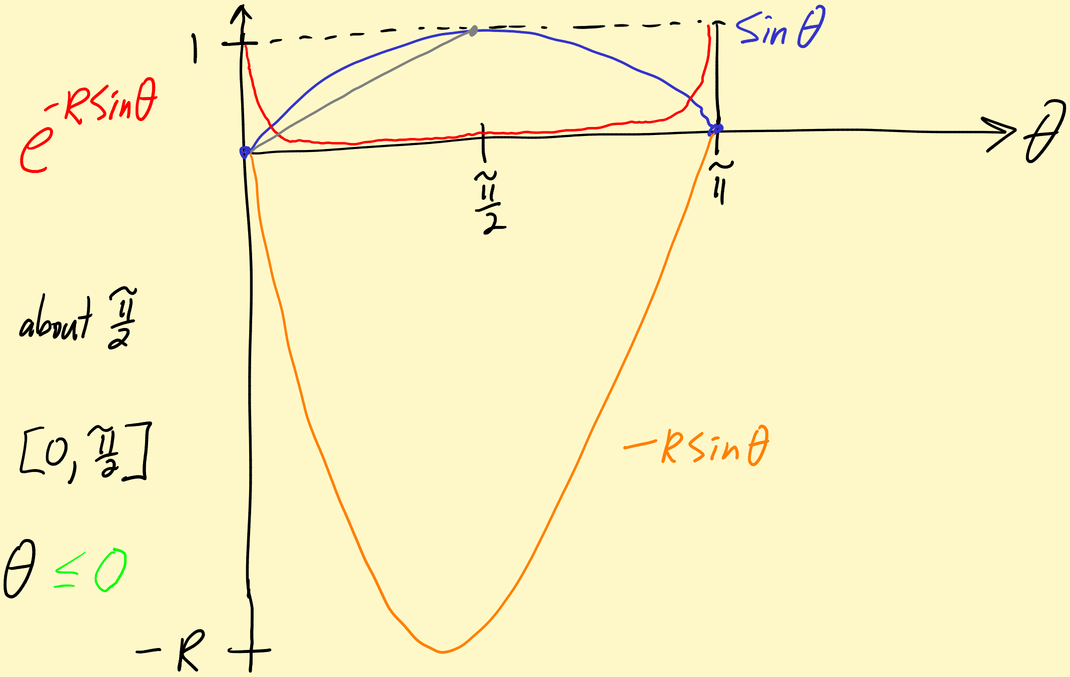
$$= i\int_0^{\pi} e^{iR\cos\theta}\, e^{-R\sin\theta}\, d\theta$$

Aha! $\left| \int_{C_R} \right| \le \int_0^{\pi} 1 \cdot e^{-R\sin\theta}\, d\theta$

$1 = |e^{iR\cos\theta}|$

Does $\int_0^{\pi} e^{-R\sin\theta}\, d\theta \to 0$ as $R \to \infty$?

Jordan: Yes!



Notice: Symmetric about $\frac{\pi}{2}$

$0 \le \frac{2}{\pi}\theta \le \sin\theta \quad [0, \frac{\pi}{2}]$

$-R\sin\theta \le -R\frac{2}{\pi}\theta \le 0$

e^x increasing.

$$\boxed{e^{-R\sin\theta} \le e^{-R\frac{2}{\pi}\theta}}$$

$$\left|\int_{C_R}\right| \le \int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta$$

$$\le \underbrace{2\int_0^{\pi/2} e^{-R\frac{2}{\pi}\theta}\,d\theta}$$

$$2\left[-\frac{\pi}{2R}e^{-R\frac{2}{\pi}\theta}\right]_0^{\pi/2}$$

$$\frac{\pi}{R}\left[-e^{-R}+1\right]$$

$$(\underbrace{1-e^{-R}}_{<1})\frac{\pi}{R} < \frac{\pi}{R}$$

$$\left|\int_{C_R}\right| < \frac{\pi}{R} \to 0$$

as $R \to \infty$.

Conclusion: $\left(\underbrace{\int_{L_1} + \int_{L_2}} + \int_{-C_\varepsilon} + \int_{C_R} \right) \frac{e^{iz}}{z}\, dz = 0$

Cauchy thm

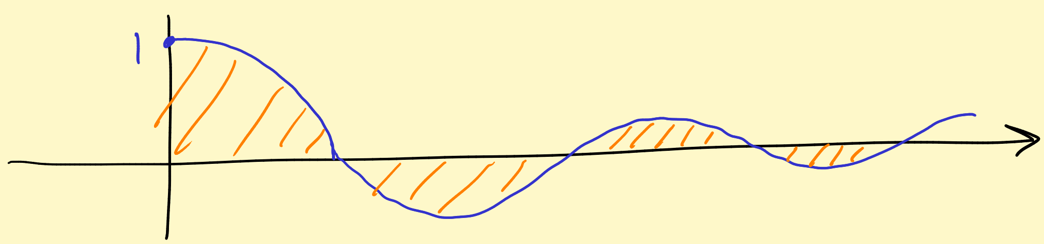
Jordan

$\varepsilon \to 0$
$R \to \infty$

$$2i \int_0^\infty \frac{\sin t}{t}\, dt + (-i\pi) + 0 = 0$$

So $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$, $\displaystyle\int_{-\infty}^\infty \frac{\sin t}{t}\, dt = \pi$

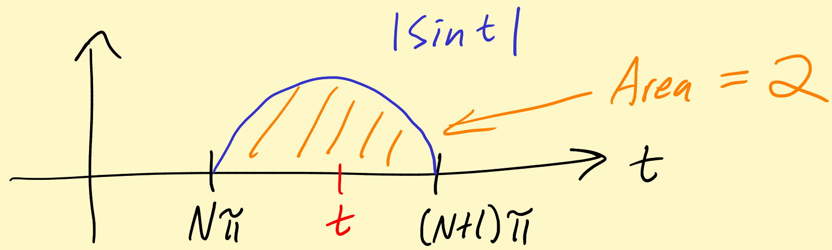
Famous: $\frac{\sin t}{t}$ is <u>not</u> absolutely integrable. $\int$ is "conditionally" convergent.



Areas of humps decrease $\searrow 0$.

Alternating series test: $\displaystyle\lim_{R \to \infty} \int_0^R \frac{\sin t}{t}\, dt$ <u>exists</u>.

But



$$\frac{|\sin t|}{(N+1)\pi} \leq \frac{|\sin t|}{t} \leq \frac{|\sin t|}{N\pi}$$

$$\int_{N\pi}^{(N+1)\pi} dt : \quad \frac{2}{(N+1)\pi} \leq \int_{N\pi}^{(N+1)\pi} \frac{|\sin t|}{t}\, dt \leq \frac{2}{N\pi}$$

$$\int_0^{(N+1)\pi} \frac{|\sin t|}{t}\, dt \geq \frac{2}{\pi}\Big(\underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N+1}}_{\substack{\text{Harmonic series} \\ \to \infty \text{ as } N \to \infty.}}\Big)$$

Jordan's lemma idea yields $\quad P(z), Q(z)$ polys

$$\deg Q \geq \deg P + 1 \longleftarrow \text{Jordan: } +1 \text{ instead of } 2$$

No zeroes of Q on $\mathbb{R}$.

$$\int_{-\infty}^{\infty} e^{iz} \frac{P(z)}{Q(z)}\, dz = 2\pi i \sum_{a \in \mathcal{O}^+} \operatorname{Res}_a e^{iz} \frac{P(z)}{Q(z)}$$

$$\mathcal{O}^+ = \{\text{zeroes of } Q \text{ in UHP}\}$$

Pf: New twist: To show $\displaystyle\int_{C_R} e^{iz} \frac{P(z)}{Q(z)}\, dz,$

use Basic Poly estimate to get an R_0 and $C>0$ such that $\left|\frac{zP(z)}{Q(z)}\right| \le C$ when $|z|=R>R_0$.

Do Jordan's thing.

Principal value integral:

$$PV\int_{-\infty}^{\infty} h(t)\,dt = \lim_{R\to\infty}\int_{-R}^{R} h(t)\,dt$$

When h is absolutely integrable:

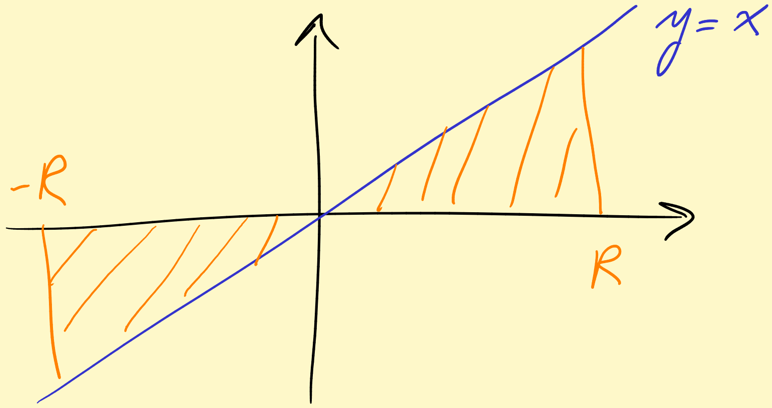
$$PV = \lim_{\substack{A\to-\infty\\ B\to\infty}}\int_A^B h(t)\,dt = \underbrace{\int_{-\infty}^{\infty} h(t)\,dt}_{\text{signed area under graph}}$$

EX: $\frac{e^{it}}{t^2+1}$ is abs integrable.

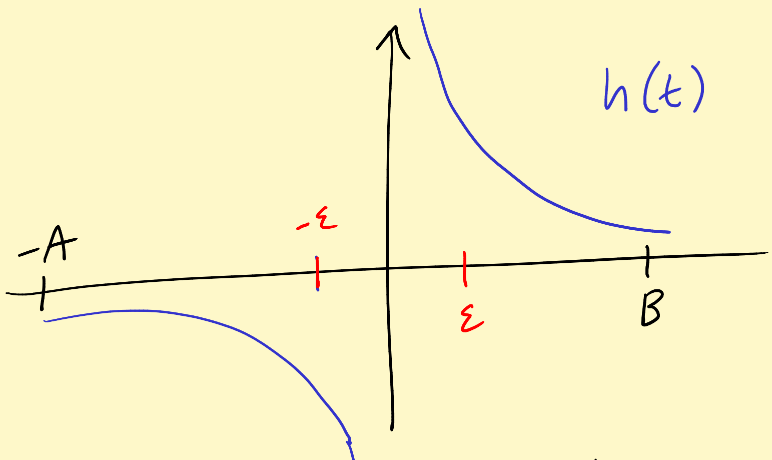
$$\left|\frac{e^{it}}{t^2+1}\right| \le \frac{1}{t^2+1} \qquad \int_{-\infty}^{\infty}\frac{1}{t^2+1}\,dt = 2\lim_{t\to\infty}\left(\text{ArcTan}\,t - 0\right)$$

$$= 2\,\frac{\pi}{2} = \pi$$

EX:



$$PV \int_{-\infty}^{\infty} x \, dx = \lim_{R \to \infty} \int_{-R}^{R} x \, dx = 0$$



$$PV \int_{-A}^{B} h(t)\,dt = \lim_{\varepsilon \to 0} \left(\int_{-A}^{-\varepsilon} + \int_{\varepsilon}^{B} \right) h(t)\,dt$$

EX: $\dfrac{1}{x^{1/2}(x+1)} \leftarrow \displaystyle\int_0^{\varepsilon} \frac{1}{\sqrt{x}}\,dx < \infty$ Safe!

$x \to \infty$ side: $\dfrac{1}{x^{1/2}(x+1)} < \dfrac{1}{x^{3/2}}$. $\displaystyle\int_1^{\infty} \frac{1}{x^{3/2}}\,dx < \infty$

Safe!