Lesson 32 More fun with the residue theorem

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| \leq \int_0^R e^{-R\sin t}\, dt < \frac{\pi}{R} \to 0 \text{ as } R \to \infty$$

$$\left| \frac{e^{iz}}{z}\, dz \right| \leq e^{-R\sin t}\, dt \qquad z(t) = Re^{it},\ z'(t) = iRe^{it}$$

Jordan's lemma also yields thm

$$\int_{-\infty}^{\infty} e^{iz} \frac{P(z)}{Q(z)}\, dz = 2\pi i \sum_{a \in \mathcal{O}^+} \operatorname{Res}_a \left(e^{iz} \frac{P(z)}{Q(z)} \right)$$

when 1) Q has no zeroes on $\mathbb{R}$

2) $\mathcal{O}^+$ = set of zeroes of Q in UHP

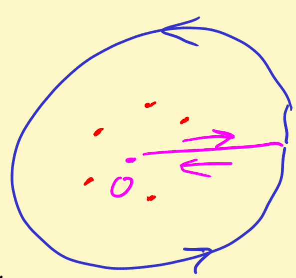
3) New case: $\deg Q = \deg P + 1$ [we did $\geq \deg P + 2$ earlier]

Pf: $\deg zP(z) = \deg Q(z)$

Basic poly estimates yield $\left| \frac{zP(z)}{Q(z)} \right| < M$ for $|z| > R$.

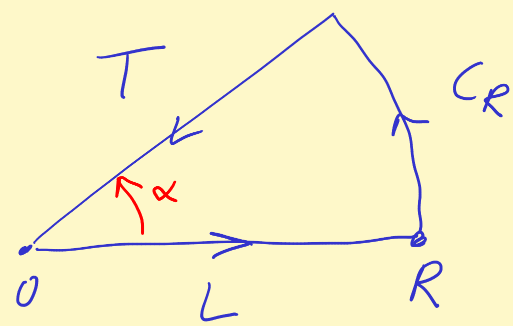
$$\left| e^{iz} \frac{P(z)}{Q(z)}\, dz \right| = \left| \frac{e^{iz}}{z} \frac{zP(z)}{Q(z)}\, dz \right| < e^{-R\sin t} M\, dt$$

EX: $I = \int_0^\infty \frac{1}{x^5+1}\, dx$



Pacman contour fails. Integrals cancel

Idea



$f(z) = \frac{1}{z^5+1}$

$$-T : z(t) = te^{i\alpha}, \quad 0 \le t \le R \qquad z'(t) = e^{i\alpha}$$

Hmmm. $$\int_T = -\int_{-T} = -\int_0^R \frac{1}{\underbrace{(te^{i\alpha})^5}+1} e^{i\alpha}\, dt$$

Aha! How wonderful if $(e^{i\alpha})^5 = 1$,

then $= -e^{i\alpha} \int_0^R \frac{1}{t^5+1}\, dt$

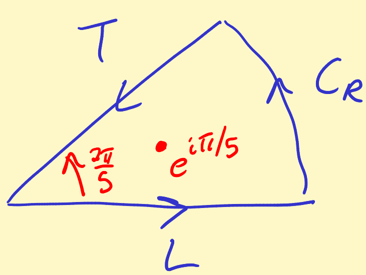
want!

Easiest α: $\alpha = \frac{2\pi}{5}$

Zeroes of z^5+1 : $z^5 = -1$

$z = e^{i\pi/5}, \; e^{i3\pi/5}, \; e^{i\pi} = -1, \; e^{i7\pi/5}, \; e^{i9\pi/5}$

inside



$$\left| \int_{C_R} \right| \le \underbrace{\left(\max_{C_R} \frac{1}{|z^5+1|} \right)}_{\le \frac{1}{R^5-1} \text{ when } R>1.} \left(\frac{2\pi}{5} R \right) \longrightarrow 0 \text{ as } R \to \infty.$$

<u>or</u> use Basic poly estimate

$$a|z|^5 \le |z^5+1| \le A|z|^5 \quad \text{when } |z|>R_0$$

Get $\left|\int_{C_R}\right| \le \frac{1}{aR^5}\left(\frac{2\pi}{5}R\right)$ when $R>R_0$

$\longrightarrow 0$ as $R\to\infty$.

$$\left(\int_L + \int_{C_R} + \int_T\right) f(z)\,dz = \underbrace{2\pi i \operatorname{Res}_{e^{i\pi/5}} f(z)}_{2\pi i \frac{1}{5(e^{i\pi/5})^4}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$R\to\infty$ $\displaystyle\int_0^\infty \frac{1}{t^5+1}\,dt + 0 + -e^{i2\pi/5}\int_0^\infty \frac{1}{t^5+1}\,dt$

$$I = \frac{2\pi i}{5e^{i4\pi/5}\left(1-e^{i2\pi/5}\right)}\,\frac{\left(-e^{-i\pi/5}\right)}{\left(-e^{-i\pi/5}\right)}$$

$$= \frac{\pi}{5}\,\frac{2i}{e^{i\pi/5}-e^{-i\pi/5}}\cdot\underbrace{\frac{-e^{-i\pi/5}}{e^{i4\pi/5}}}_{=-\frac{1}{e^{i5\pi/5}}=\frac{-1}{e^{i\pi}}=\frac{-1}{-1}=1}$$

$$= \frac{\pi/5}{\sin \pi/5}$$

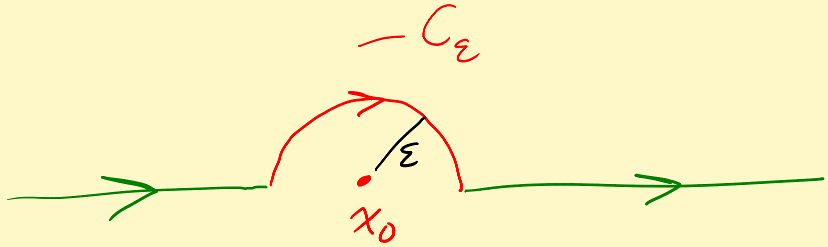
$$\operatorname{Res}_{e^{i\pi/5}} \frac{1 \;\leftarrow g(z)}{z^5+1 \;\leftarrow h(z)} \;:\quad h(e^{i\pi/5}) = 0 \;\leftarrow \text{simple zero}$$

$$h'(e^{i\pi/5}) = 5(e^{i\pi/5})^4 \neq 0$$

$$\operatorname{Res}_a \frac{g}{h} = \frac{g(a)}{h'(a)} \;\checkmark$$

[Don't factor z^5+1 !]

General fact: If $f(z)$ has a simple pole at $x_0 \in \mathbb{R}$, then

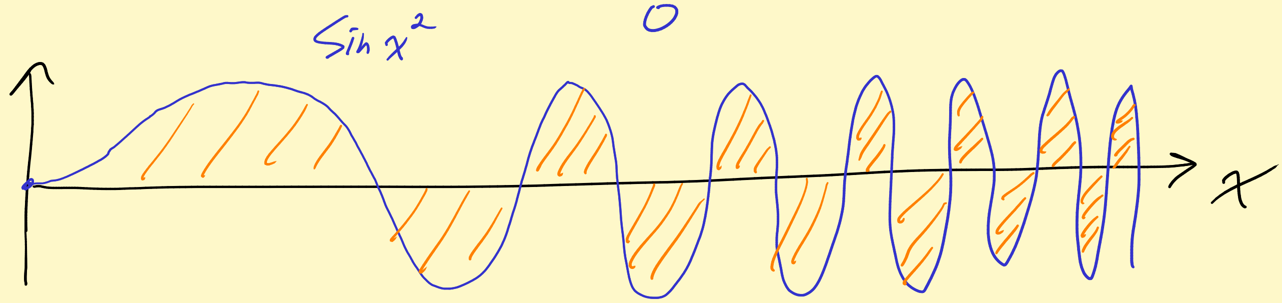


$$\lim_{\varepsilon\to 0} \int_{-C_\varepsilon} f\,dz = \underset{\substack{\uparrow \\ \text{clockwise}}}{-} \underset{\substack{\uparrow \\ \text{half circle}}}{\left(\frac{1}{2}\right)} \left[2\pi i \operatorname{Res}_a f(z)\right]$$

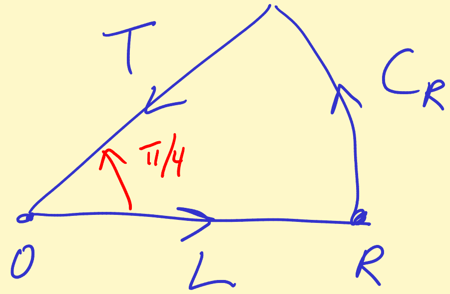
Pf: $f(z) = \dfrac{A_0}{z-x_0} + \underbrace{H(z)}_{\text{power series near } x_0.}$ Bounded near x_0.

$$\int_{-C_\varepsilon} f(z)\,dz = \underbrace{\int_{-C_\varepsilon} \frac{A_0}{z-x_0}\,dz}_{= -\frac{1}{2} 2\pi i A_0 \;\checkmark} + \underbrace{\int_{-C_\varepsilon} H(z)\,dz}_{|S| \le M(\pi\varepsilon) \to 0 \text{ as } \varepsilon\to 0}$$

EX: Fresnel integrals $\int_0^\infty \sin x^2\,dx = \frac{\sqrt{2\pi}}{4}$



Areas of humps $\searrow 0$. $\lim_{R\to\infty}\int_0^R$ exists via Alt series test.



$$f(z) = e^{-z^2}$$

$$\int_L e^{-z^2}\,dz = \int_0^R e^{-t^2}\,dt \longrightarrow \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

$$\left|\int_{C_R} e^{-z^2}\,dz\right| = \left|\int_0^{\pi/4} e^{-[Re^{it}]^2}\, iRe^{it}\,dt\right|$$

$$\leq \int_0^{\pi/4} e^{-R^2\cos 2t}\,dt$$

Jordan's lemma idea shows $\to 0$ as $R\to\infty$.

$$\int_T = -\int_{-T} = -\int_0^R e^{-[te^{i\pi/4}]^2}\, e^{i\pi/4}\,dt$$

$$\left(e^{i\pi/4}\right)^2 = e^{i\pi/2} = i$$

$$\int_T = -\int_0^R e^{-it^2} e^{i\pi/4}\, dt = -e^{i\pi/4}\int_0^R \cos t^2 - i \sin t^2\, dt$$

Fresnel!

Last thing: Cauchy's thm: $\left(\int_L + \int_{C_R} + \int_T\right) e^{-z^2} dz = 0$

$R \to \infty \qquad \frac{\sqrt{\pi}}{2} + 0 - e^{i\pi/4}\left[\int_0^\infty \cos t^2\, dt - i\int_0^\infty \sin t^2\, dt\right] = 0$

$$\int_0^\infty \cos t^2\, dt - i\int_0^\infty \sin t^2\, dt = \frac{\sqrt{\pi}/2}{e^{i\pi/4}} = \frac{\sqrt{\pi}}{2} e^{-i\pi/4} = \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}$$

See $\int_0^\infty \cos t^2\, dt = \int_0^\infty \sin t^2\, dt = \frac{\sqrt{2\pi}}{4}$.

Hard part: Choosing right contour!