# Lesson 33 The Argument principle and Rouché's theorem

No in-person class on Monday, April 8 (eclipse day), but I will post a recorded lecture & notes in the AFTERMATH for that day.

Last homework: HWK 10 posted, due FRIDAY, April 12.

Parallel facts about zeroes and poles

| Zeroes | Poles |
|---|---|
| $f(z)$ has a zero of order $N$ at $z_0$ | $f(z)$ has a pole of order $N$ at $z_0$ |
| $f(z) = a_N (z-z_0)^N + \cdots$ | $f(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{A_1}{(z-z_0)} + \text{power series}$ |
| $= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots\right]}_{F(z)}$, $F(z_0) = a_N \neq 0$ | $= \frac{1}{(z-z_0)^N}\left[\underbrace{A_N + \text{power series}}_{F(z)}\right]$, $F(z_0) = A_N \neq 0$ |
| $f(z) = (z-z_0)^N F(z)$ | $f(z) = (z-z_0)^{-N} F(z)$ |

Let $n = \pm N$. $\quad f(z) = (z-z_0)^n F(z)$

$F$ analytic, $F(z_0) \neq 0$.

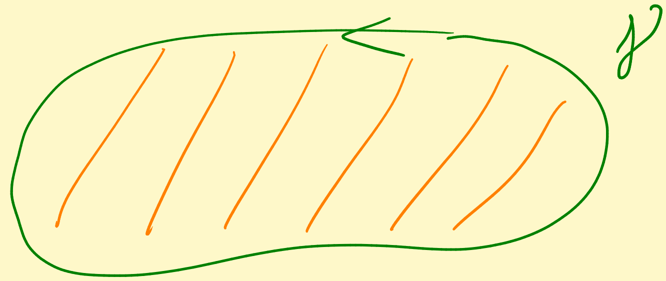
Then
$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}F(z) + (z-z_0)^n F'(z)}{(z-z_0)^n F(z)}$$

$$= \frac{n}{z-z_0} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic about } z_0}$$

$z_0$ is a simple pole of $\frac{f'}{f}$

Fact: $\operatorname{Res}_{z_0} \frac{f'}{f} = n = \begin{cases} \text{order of zero at } z_0 \text{ if } f \text{ has a zero} \\ -\text{order of pole at } z_0 \text{ if } f \text{ has a pole} \end{cases}$

## Argument principle




$\gamma$ simple closed curve.

**Case zeroes only.** $f$ is analytic inside and on $\gamma$ with no zeroes on $\gamma$. Then

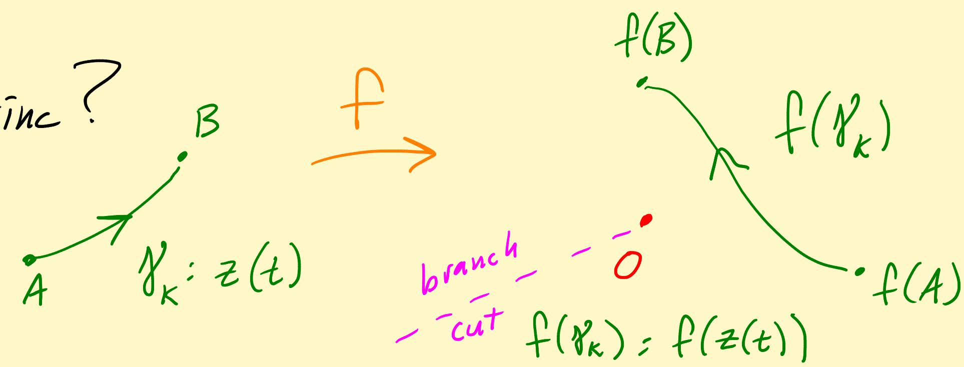
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \begin{pmatrix} \#\text{ zeroes of } f \\ \text{inside } \gamma \\ \text{counted with multiplicity} \end{pmatrix}$$

$$\text{Pf}: \quad = \frac{1}{2\pi i}\left[2\pi i \sum_{\substack{a\in Z_f \\ \uparrow \\ \text{finite!}}} \operatorname{Res}_a \frac{f'}{f}\right] = \sum_{k=1}^{N} \underset{\substack{\uparrow \\ \text{Multiplicities} \\ \text{of zeroes.}}}{m_k}$$

**Case zeroes and poles** $f$ analytic inside and on $\gamma$ with finitely zeroes and poles inside $\gamma$, no zeroes or poles on $\gamma$.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \begin{pmatrix} \#\text{ zeroes } f \\ \text{inside } \gamma \\ \text{with mult} \end{pmatrix} - \begin{pmatrix} \#\text{ poles } f \\ \text{inside } \gamma \\ \text{with mult} = \text{order} \end{pmatrix}$$

Why "Argument" princ?




Think $\gamma_k$ short.

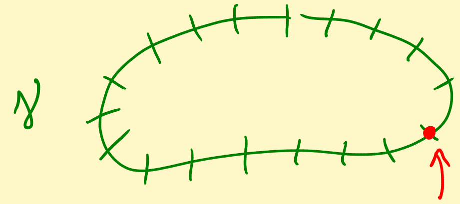
Now $$\int_{\gamma_k} \frac{f'}{f}\,dz = \int_{\gamma_k} \frac{d}{dz}\left(\log f\right) dz$$

$$= \log f(B) - \log f(A)$$

$$= \operatorname{Ln}|f(B)| + i \arg f(B) - \left(\operatorname{Ln}|f(A)| + i \arg f(A)\right)$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right) + i\, \Delta_{\gamma_k} \arg f$$

does not depend on choice of branch




pairwise cancelation

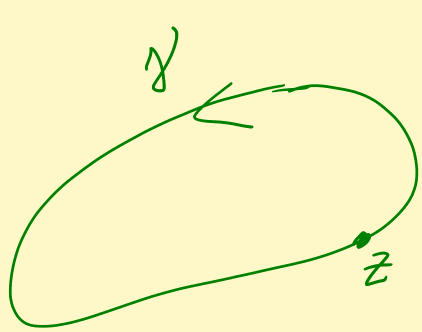
first $-\operatorname{Ln}|f(z_0)|$ cancels last $\operatorname{Ln}|f(z_N)|$

$$\sum \int_{\gamma_k} \frac{f'}{f}\,dz$$

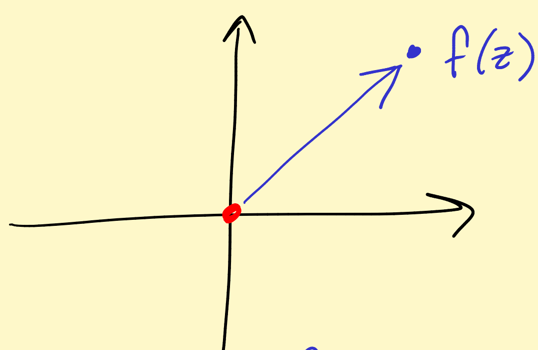
$$= 0 + i \sum \Delta_{\gamma_k} \arg f$$

$$\int_{\gamma} \frac{f'}{f}\,dz = i\, \Delta_{\gamma} \arg f$$

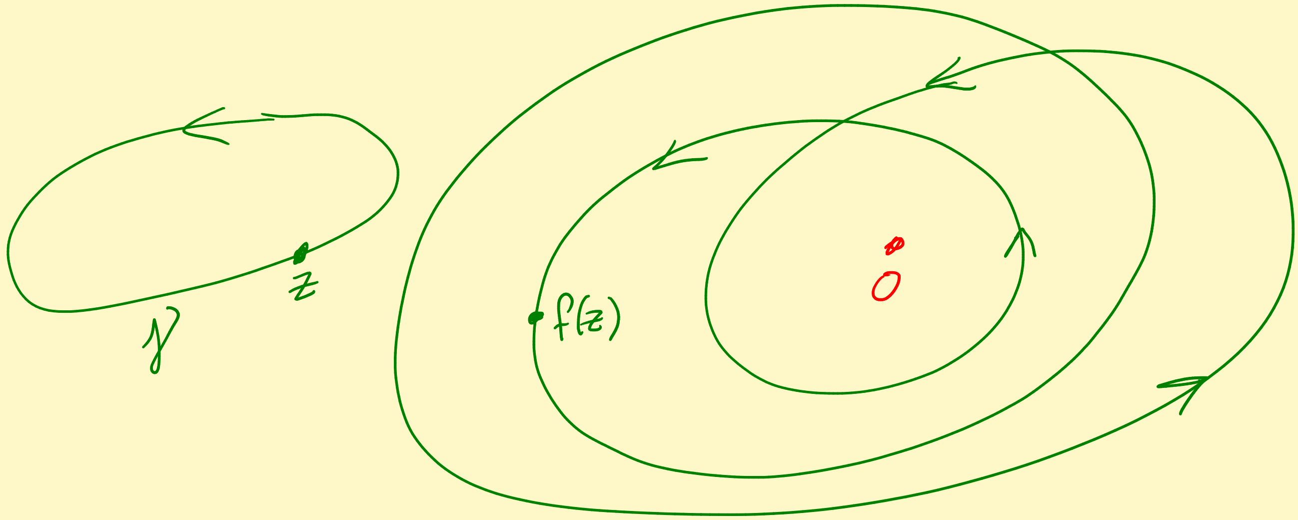
Aha! $$\frac{1}{2\pi i}\int_{\gamma} \frac{f'}{f}\,dz = \frac{1}{2\pi}\Delta_{\gamma} \arg f = \left(\begin{array}{c}\text{\# times } f(z) \\ \text{spins around} \\ 0 \\ \text{as } z \text{ goes around} \\ \gamma\end{array}\right)$$




z goes around $\gamma$ counterclockwise




Total times f goes around 0 is # zeroes.

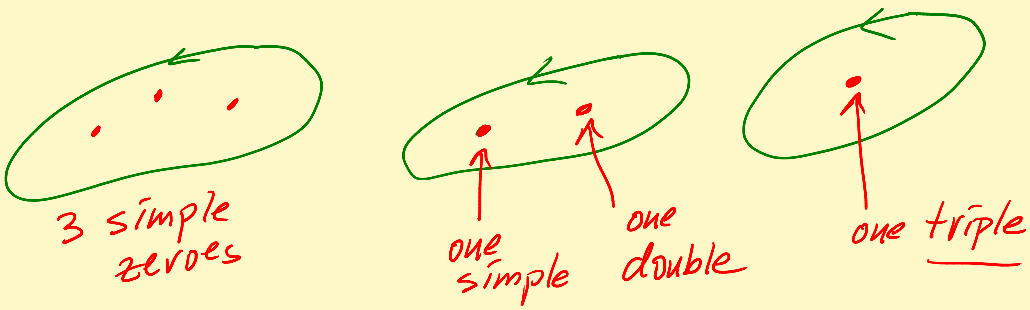



Stand at 0. Point to $f(z)$.

Spin 3 times as $z$ goes around $\gamma$.

$f(z)$ has 3 zeroes in $\gamma$.

Hmmm.



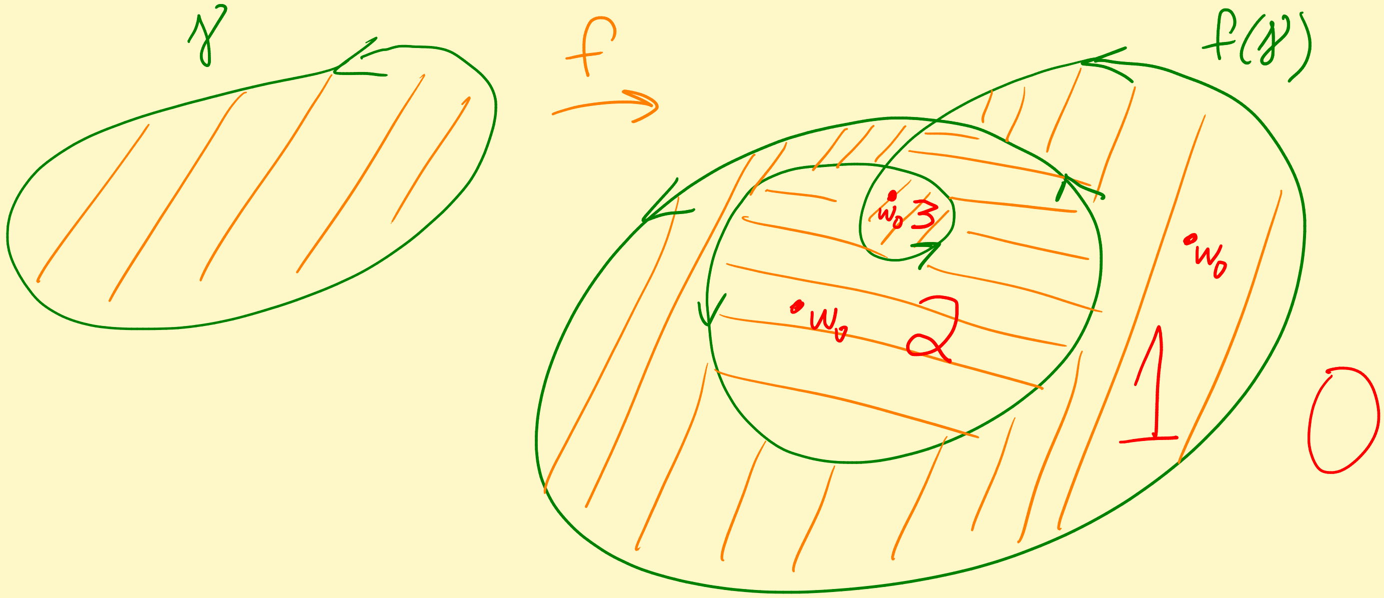

Nice observation: Let $F(z) = f(z) - w_0$.

If $F(z_0) = f(z_0) - w_0 = 0$, zero of mult $m$.

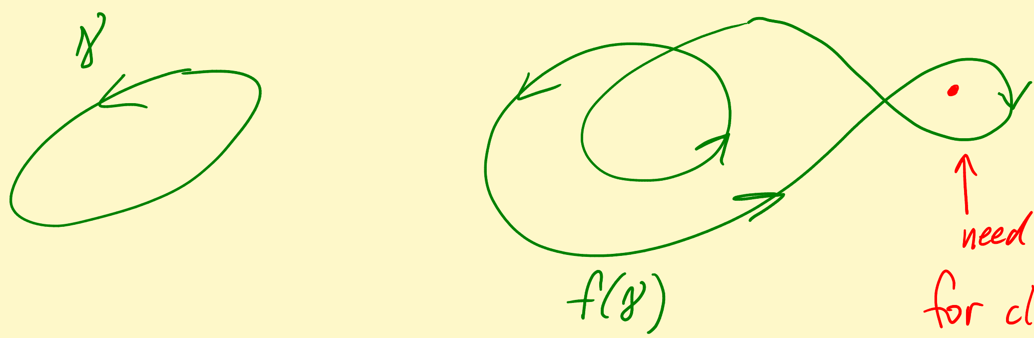
Say $f(z_0) = w_0$ with multiplicity $m$.

Think: $f(z)$ hits $w_0$ $m$ times at $z = z_0$.

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z) - w_0}$$

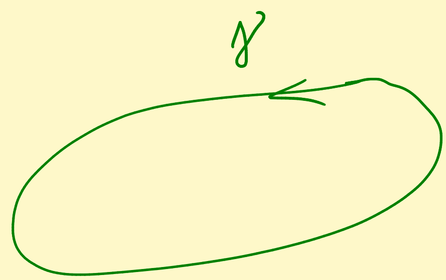



Picture?




need poles in $\gamma$ for clockwise loops!

## Rouché's theorem




$f$ and $h$ analytic inside and on a simple closed curve $\gamma$.

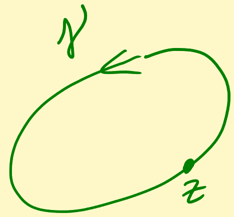
If $|h(z)| < |f(z)|$ along $\gamma$, then $f$ and $f+h$ have same # zeroes inside $\gamma$ counted with multiplicity.

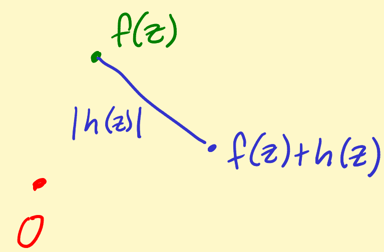
"Pf": $f(z) = $ me

$f(z)+h(z) = $ my dog

$|h(z)| = $ variable leash length

telephone pole $= 0$




If leash length < dist from me to pole, dog goes around pole same # times as I do!

Arg Princ. ✓