Lesson 34 Fun with Fibonacci numbers

$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_5 = 5,\ \ldots$

$a_0 = 1,\ a_1 = 1. \qquad a_n = a_{n-1} + a_{n-2} \quad n \geq 2$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

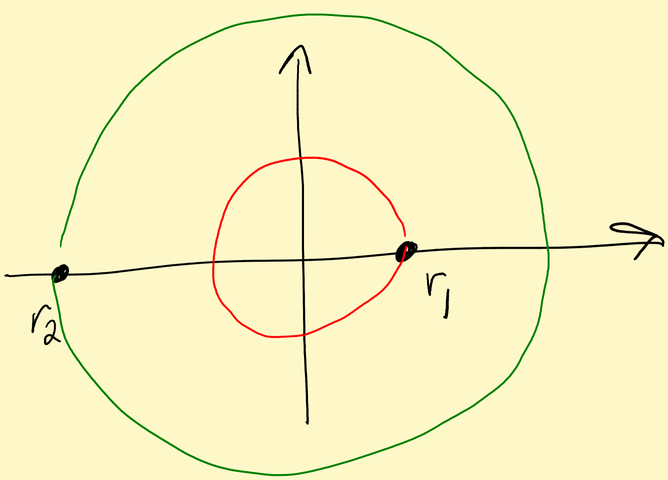
Assume R of C is $R > 0$.

$$f(z) = 1 + z f(z) + z^2 f(z)$$

Get $f(z) = \dfrac{1}{1-z-z^2} = \dfrac{-1}{z^2+z-1}$

Zeroes z^2+z-1 : $\quad r_1, r_2 = \dfrac{-1 \pm \sqrt{5}}{2}$

Take $r_1 = \dfrac{-1+\sqrt{5}}{2}$, $r_2 = \dfrac{-1-\sqrt{5}}{2}$



Rational fcn $R(z) = \dfrac{-1}{z^2+z-1}$ $\qquad$ R of C $= r_1$

It has a power series $\sum_0^\infty b_n z^n$. R of C $= r_1$

$$R(z) = 1 + zR(z) + z^2 R(z)$$

Aha! Coeff b_n satisfy recursion formula for the Fib #s! So b_n's $= a_n$'s !

Today: $$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

$$f(z) = \frac{-1}{z^2+z-1} = \frac{-1}{(z-r_1)(z-r_2)}$$

$$= \frac{A}{z-r_1} + \frac{B}{z-r_2} \quad \text{(Part. frac.)}$$

$$-1 = A(z-r_2) + B(z-r_1)$$

$$= \underbrace{(A+B)}_{0} z + \underbrace{(-r_2 A - r_1 B)}_{-1}$$

$$\boxed{B = -A}$$

$$-r_2 A - r_1 \underbrace{(-A)}_{B} = -1$$

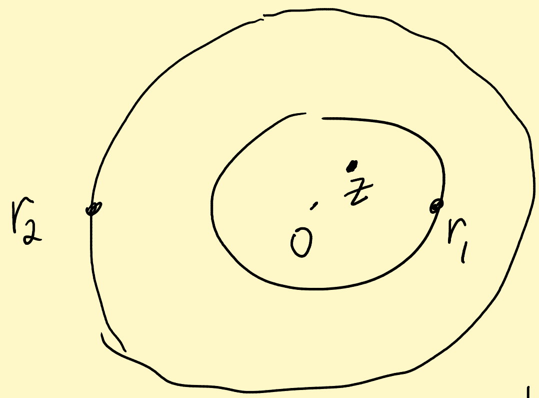
$$A = \frac{-1}{r_1 - r_2}$$

$$= \frac{-1}{\left(\frac{-1+\sqrt{5}}{2}\right) - \left(\frac{-1-\sqrt{5}}{2}\right)}$$

$$B = -A = \frac{1}{\sqrt{5}} \qquad \boxed{A = \frac{-1}{\sqrt{5}}}$$

$$f(z) = \frac{1}{\sqrt{5}}\left[\frac{1}{z-r_2} - \frac{1}{z-r_1}\right]$$

$$\frac{1}{z-r_1}\cdot\frac{\left(\frac{-1}{r_1}\right)}{\left(\frac{-1}{r_1}\right)} = \frac{-1/r_1}{1-\frac{z}{r_1}}$$



$$\frac{|z|}{r_1} < 1.$$

$$-\frac{1}{r_1}\left[1 + \left(\frac{z}{r_1}\right) + \left(\frac{z}{r_1}\right)^2 + \cdots\right]$$

Do same thing for $\frac{1}{z-r_2}$ term:

$$f(z) = \frac{1}{\sqrt{5}}\left[-\frac{1}{r_2}\left(1+\left(\frac{z}{r_2}\right)+\left(\frac{z}{r_2}\right)^2+\cdots\right) + \frac{1}{r_1}\left(1+\left(\frac{z}{r_1}\right)+\cdots\right)\right]$$

$$= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{1}{r_1^{n+1}} - \frac{1}{r_2^{n+1}}\right)z^n \qquad \underline{a_n = \frac{1}{\sqrt{5}}\left(\frac{1}{r_1^{n+1}} - \frac{1}{r_2^{n+1}}\right)}$$

see p.7 for algebra

Know that $f(z)$ has a Laurent expansion in annulus $A(r_1, |r_2|)$.

Nice trick to find coeff:

$$f(z) = \frac{1}{\sqrt{5}}\Big(\underbrace{\frac{1}{z-r_2}} - \underbrace{\frac{1}{z-r_1}} \Big) \qquad r_1 < |z| < |r_2|$$

$$\frac{1}{z-r_2} \cdot \frac{\left(\frac{-1}{r_2}\right)}{\left(\frac{-1}{r_2}\right)} \qquad\qquad \frac{1}{z-r_1}\cdot\frac{\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)} = \frac{1/z}{1-\frac{r_1}{z}}$$

$|\cdot| < 1$ ✓

$$\frac{-1/r_2}{1-\frac{z}{r_2}}$$

$|\cdot| < 1$

$$= \frac{1}{z}\left(1 + \left(\frac{r_1}{z}\right) + \left(\frac{r_1}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z} + \frac{r_1}{z^2} + \frac{r_1^2}{z^3} + \cdots$$

= Geom series

positive terms in Laurent. | negative terms in Laurent expansion!

Laurent expansion outside $D_{|r_2|}(0)$:

$$\frac{1}{z-r_2}\cdot\frac{\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)} = \frac{1/z}{1-\frac{r_2}{z}} \qquad \text{etc.}$$

<u>Remark</u>: Can use same methods to get Laurent for rational fcns, but need:

Partial fracs: $\dfrac{A}{z-r}, \quad \dfrac{A}{(z-r)^2}, \quad \dfrac{A}{(z-r)^3}, \quad \ldots$

$\checkmark$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad |z| < 1$$

Diff: $$\frac{1}{(1-z)^2} = z + 2z + 3z^2 + \cdots$$

$$\frac{A}{(z-r)^2} = \frac{A}{(z-r)^2} \frac{\left(\frac{1}{r^2}\right)}{\left(\frac{1}{r^2}\right)} = \frac{A/r^2}{\left(1-\frac{z}{r}\right)^2}$$

use if $|z| < r$

$$= \frac{A}{(z-r)^2} \frac{\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)} = \frac{A\left(\frac{1}{z}\right)}{\left(1-\frac{r}{z}\right)^2} \quad \text{if } r < |z|$$

Get negative terms of a Laurent expansion...

$$\frac{2}{(1-z)^3} = 2 + 3\cdot 2z + 4\cdot 3z^2 + \cdots \qquad \text{etc.}$$

<u>Fact</u>: Can multiply power series like polynomials.

$$f(z) = \sum_0^\infty a_n z^n, \quad g(z) = \sum_0^\infty b_n z^n$$

$$h(z) = f(z)\, g(z)$$

Then $h(z) = \sum_{0}^{\infty} c_n z^n$ with $R_h \geq \min(R_f, R_g)$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$
$$= \sum_{k=0}^{n} a_k b_{n-k}$$

$$k + (n-k) = n$$

$$(a_k z^k)(b_{n-k} z^{n-k}) = a_k b_{n-k} z^n$$

<u>Pf</u>: Taylor's formula plus Leibnitz's formula.

Leibnitz: $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$

Taylor's: $c_n = \dfrac{(fg)^{(n)}(0)}{n!} = \sum_{k=0}^{n} \left(\dfrac{n!}{\underbrace{k!(n-k)!}_{n!}} \right) f^{(k)}(0)\, g^{(n-k)}(0)$

$$= \sum_{k=0}^{n} \underbrace{\frac{f^{(k)}(0)}{k!}}_{a_k} \underbrace{\frac{g^{(n-k)}(0)}{(n-k)!}}_{b_{n-k}}$$

✓

Algebra: $\dfrac{1}{r_1^{n+1}} - \dfrac{1}{r_2^{n+1}} = \dfrac{r_2^{n+1} - r_1^{n+1}}{(r_1 r_2)^{n+1}}$

$$r_1 r_2 = \frac{-1+\sqrt{5}}{2} \cdot \frac{-1-\sqrt{5}}{2} = \frac{-4}{4} = -1$$

$z^2 - z - 1 = (z-r_1)(z-r_2)$
$= z^2 - (r_1+r_2)z + \underbrace{r_1 r_2}_{=-1}$

$$a_n = \frac{1}{\sqrt{5}}\left[\frac{r_2^{n+1}}{(-1)^{n+1}} - \frac{r_1^{n+1}}{(-1)^{n+1}}\right]$$

$$= \frac{1}{\sqrt{5}}\left[(-r_2)^{n+1} - (-r_1)^{n+1}\right]$$

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \checkmark$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}\left[1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1}\right]$$

this is $|\cdot| < 1$, so $\to 0$ as $n \to \infty$

asymptotic growth of a_n as $n \to \infty$

Could use ratio test now!