Lesson 35 Rouché's theorem

HWK 10 due FRIDAY

Rouché's Suppose f and h are analytic inside and on a simple closed curve γ and $|h(z)| < |f(z)|$ on γ. Then f and $f+h$ have the same number of zeroes inside γ counted with multiplicity.

EX: $\overbrace{\underbrace{z^3 + 9z}_{h(z)} + \underbrace{27}_{f(z)}}^{f+h}$ on $D_2(0)$. How many zeroes.

$h(z)$ leash length

big one $= f(z)$ me

$f(z) + h(z)$ dog

$0 =$ telephone pole

Note: $|h(z)| \le |z|^3 + 9|z| \le \underbrace{2^3 + 9\cdot 2}_{26}$ on $C_2(0)$

$< 27 = |f(z)|$ on $C_2(0)$

Rouché's: f and $f+h$ have same # zeroes

none. So $f+h$ has none too.

EX Show that all the zeroes of $z^6 - 5z^2 + 10$ fall in the annulus $\{z : 1 < |z| < 2\}$.

On $C_2(0)$: $\underbrace{z^6}_{\substack{\uparrow \\ \text{big one} \\ f(z)}} \underbrace{- 5z^2 + 10}_{h(z)} = f + h$

$$|h(z)| = |-5z^2+10| \le \underbrace{5\cdot 2^2 + 10}_{30} \text{ on } C_2(0)$$

$$< |z^6| = 2^6 = 64$$

$f(z)$ has $\underline{\underline{6}}$ zeroes in $C_2(0)$.

So $f+h$ also has 6.

On $C_1(0)$: $\underbrace{z^6 - 5z^2}_{h(z)} + \underbrace{10}_{\substack{\uparrow \\ \text{big one} \\ f(z)}} = f+h$

$$|h(z)| = |z^6 - 5z^2| \le \underbrace{|z|^6 + 5|z^2|}_{=} $$

$$1^6 + 5\cdot 1^2 = 6 \text{ on } C_1(0)$$

$$< \underbrace{10}_{} = |f(z)| \text{ on } C_1(0).$$

no zeroes inside $C_1(0)$.

So $f+h$ has $\underline{\text{no}}$ zeroes in $C_1(0)$.

All 6 zeroes of $z^6 - 5z^2 + 10$ fall in $A(1,2)$.

Why Rouché's is true

$|h| < |f|$ on γ

Let $g = f + h$. Want to see f and g have same # zeroes inside γ.

$$\underbrace{|h|}_{|g-f|} < |f|$$

(*) $|g - f| < |f|$ ← some books use g instead of h.

Divide by $|f|$. [Note $|g - f| < |f|$

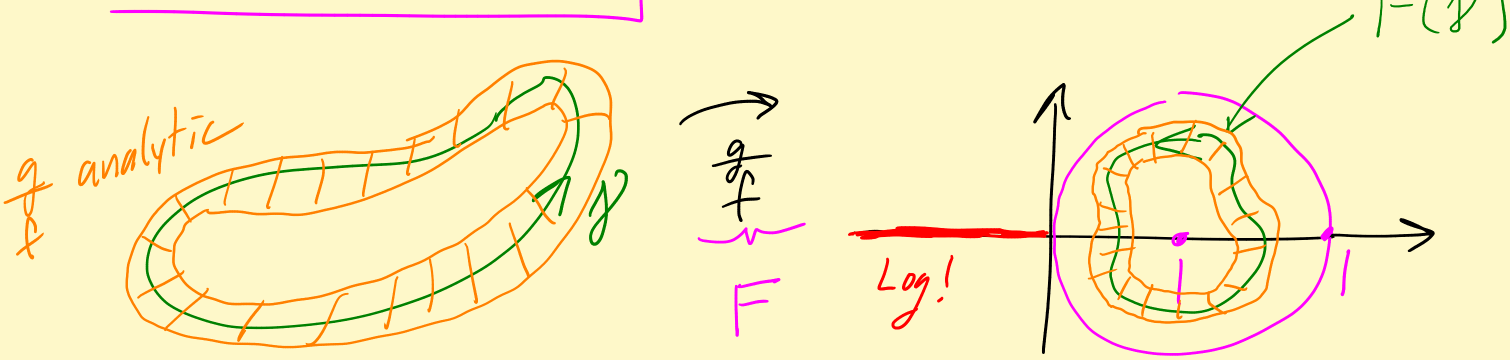
$g=0$ ⇒ $|f| < |f|$. No.

$|f| > 0$. No zeroes of f on γ

No zeroes of g on γ]

$$\left|\frac{g}{f} - 1\right| < 1$$

$$\boxed{\left|1 - \frac{g}{f}\right| < 1} \quad \text{on } \gamma$$



$$F = \frac{g}{f} \qquad \frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f}$$

Aha!

$$\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{F'}{F}\,dz}_{\frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\operatorname{Log} F(z)\,dz} = \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'}{g}\,dz}_{\#\text{ zeroes } g \text{ in } \gamma} - \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz}_{\#\text{ zeroes } f}$$

$$= \operatorname{Log} F(z)\Big|_{\text{START}}^{\text{END}}$$

$$= 0$$

<u>Open mapping theorem:</u> Non-constant analytic fcns map open sets to open sets.