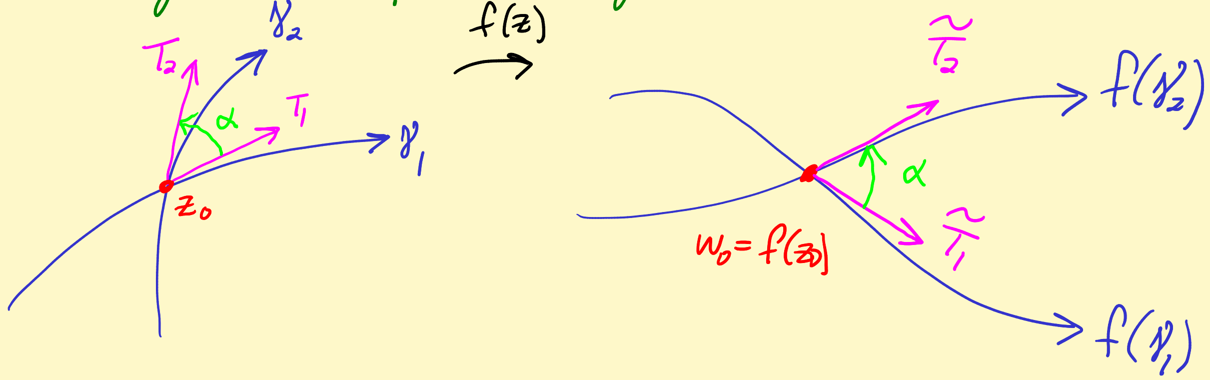
Lesson 36 Conformal mapping

Fact If f is analytic and f' is non-vanishing, then f is conformal, meaning that it preserves angles:



$\gamma_j : z_j(t), \quad z_j(0) = z_0 \qquad T_j = z_j'(0) \qquad j = 1, 2$

$f(\gamma_j) : f(z_j(t)) \qquad \tilde{T}_j = f'(\underbrace{z_j(0)}_{z_0})\, \underbrace{z_j'(0)}_{T_j} \qquad j = 1, 2$

So $\tilde{T}_j = \underbrace{f'(z_0)}_{re^{i\theta}} T_j$

$\tilde{T}_j = r e^{i\theta} T_j$ ← get complex direction $\tilde{T}_j$ by stretching T_j by a factor of $r > 0$ and rotating by θ (in counterclockwise sense)

See α is preserved and the sense is preserved.

Fun fact Reverse is true! (almost)

$F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

Jacobian matrix $\quad \mathbb{J} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$

matrix mult by $\mathbb{J}$ determines how vectors get transformed under F

Assume u, v are $\mathcal{C}^1$-smooth and

and $\det \mathbb{J} \neq 0$. Analytic geometry: linear map is a rotation if and only if

$$\mathbb{J} = R \cdot \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

See C-R equns!

<u>EX</u>: $f(z) = z^2$. $f'(z) = 2z$ non-vanishing on $\mathbb{C} - \{0\}$.

f preserves angles on $\mathbb{C} - \{0\}$.

Hmmm. At $z = 0$. $\qquad z_1(t) = e^{i\theta_1} t$

$z_2(t) = e^{i\theta_2} t$

$$f(z_j(t)) = e^{i2\theta_j} t^2$$

$$\left.\begin{aligned} \alpha &= \theta_2 - \theta_1 \\ \tilde{\alpha} &= 2(\theta_2 - \theta_1) \end{aligned}\right\} \text{ angles get doubled.}$$

(opens like a fan)

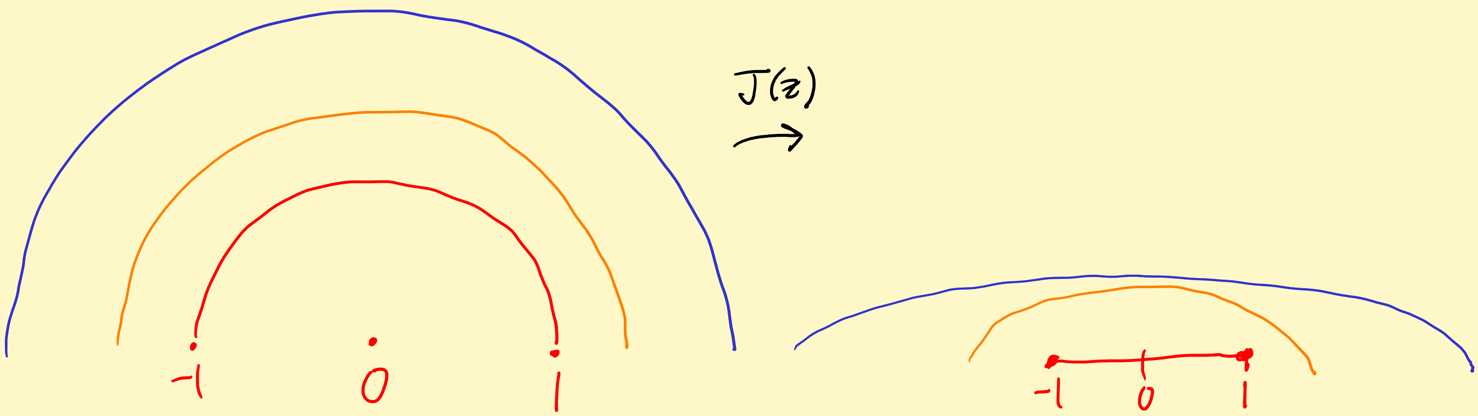
HWK 10: $\quad z^\alpha = e^{\alpha \operatorname{Log} z}$

Famous map: Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

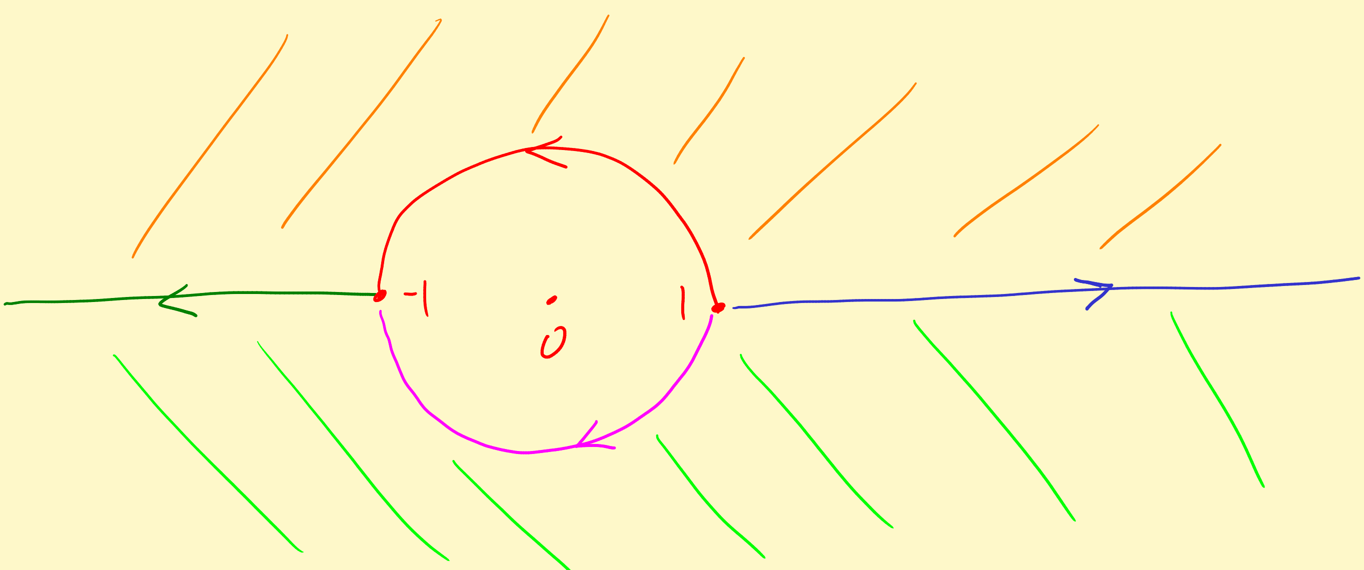
On circles: $J(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{re^{i\theta}}\right)$

$$= \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}\underbrace{e^{-i\theta}}_{\cos\theta - i\sin\theta}\right)$$

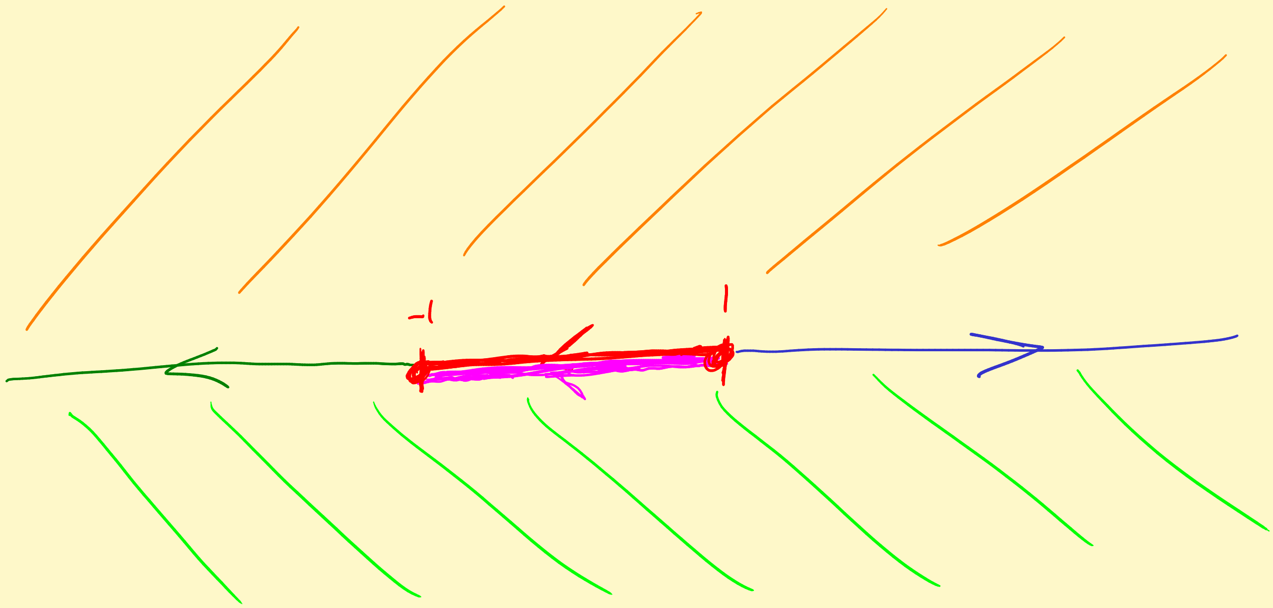
$$= \underbrace{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]\cos\theta + i\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]\sin\theta}_{\text{ellipse}}$$

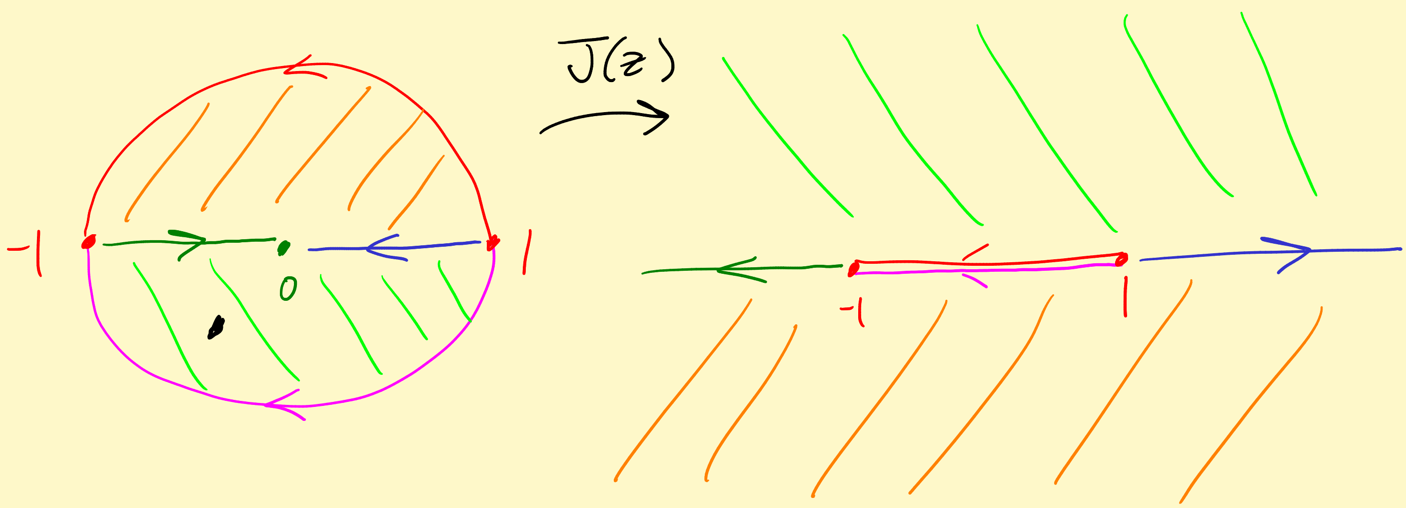


Foci at ± 1



$J(z) \longrightarrow$

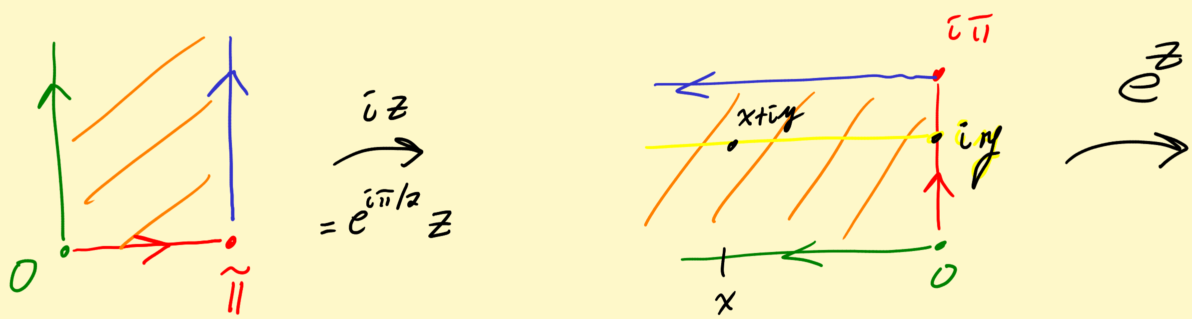




Complex cosine fcn : $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$$= \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right)$$

$$= J\left(e^{iz}\right)$$

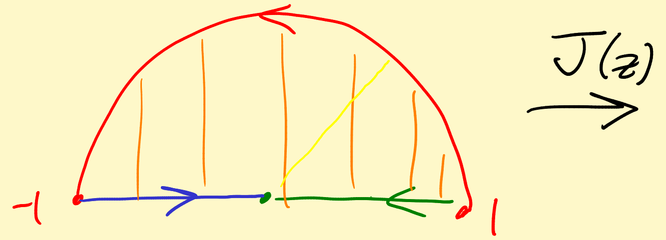


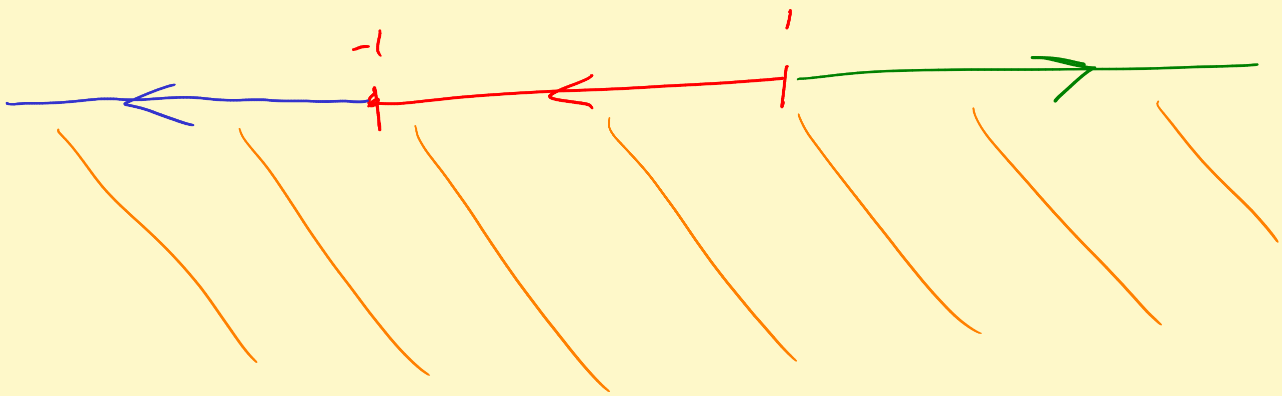
$$e^{x+iy} = e^x e^{iy}$$

$-\infty < x < 0$ $\qquad$ $0 < y < \pi$

$0 < e^x < 1$

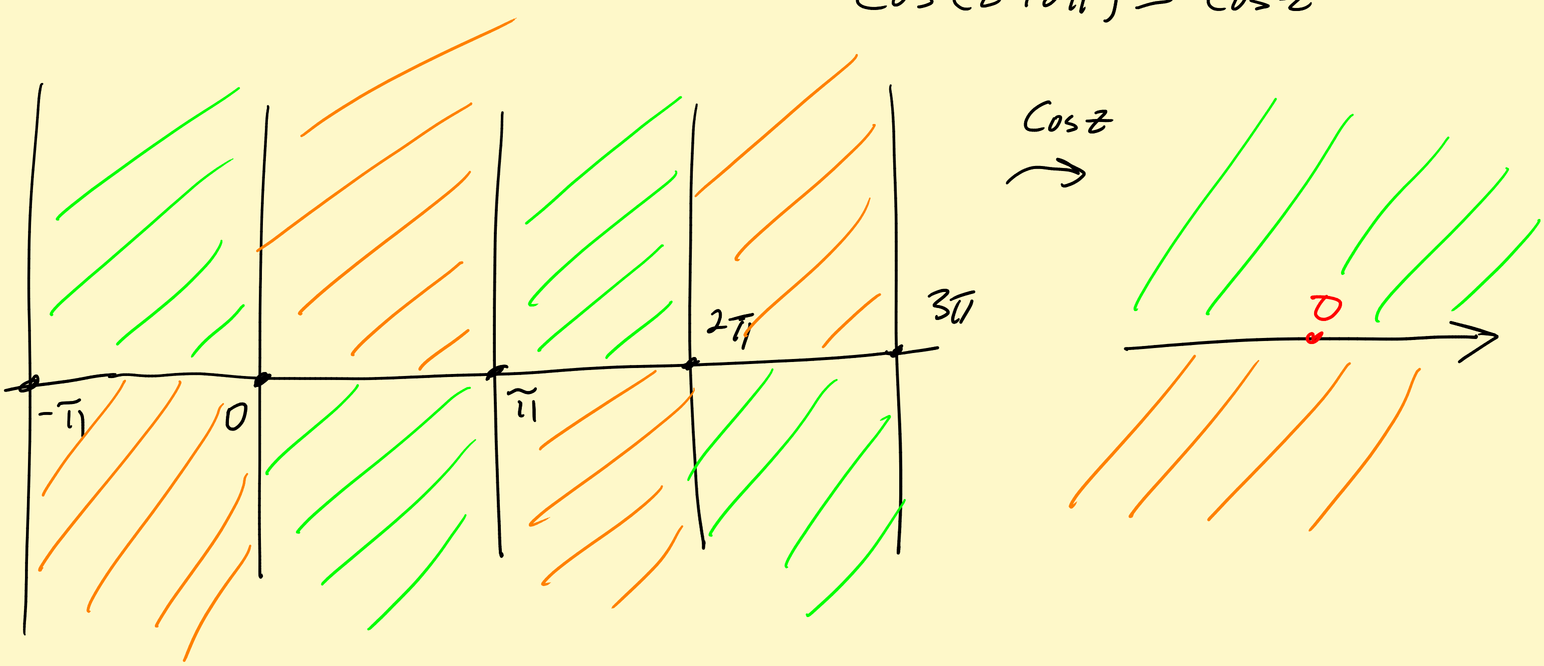
radius $\qquad$ angle





Aha! $\quad \cos(-z) = \cos(z)$ $\qquad$ $\cos(z+\pi) = -\cos z$

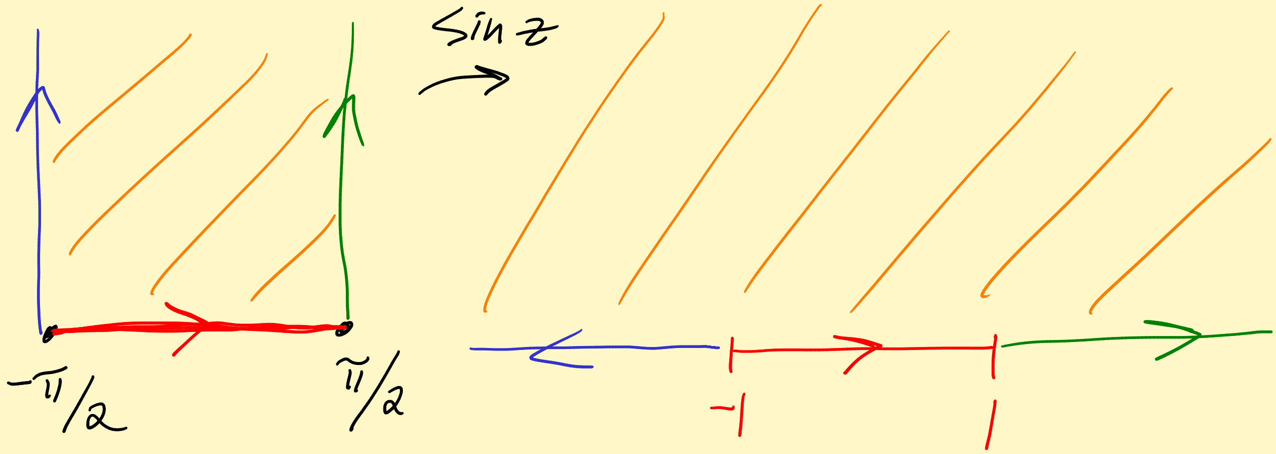
$\cos(z+2\pi) = \cos z$



Singularity of $\cos z$ at $z=\infty$ is defined to be the singularity of $\cos\frac{1}{z}$ at $z=0$.

$e^{1/z}$, $\cos\frac{1}{z}$, $\sin\frac{1}{z}$ all have <u>essential</u> singularities at 0.

<u>What about $\sin z$?</u> $\qquad \sin z = \cos\left(z - \frac{\pi}{2}\right)$



Similar checkerboard pic on $\mathbb{C}$