Lesson 37 Linear fractional transformations (LFTs)

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

Book calls LFTs Möbius transformations.

Some books: Möbius ones are $\frac{z-a}{1-\bar{a}z}$ ← 1-1, onto, analytic $D_1(0)$

Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ← det = 0

Then (one row) = multiple (other row)

$\Rightarrow$ LFT = const, or denom = 0.

LFT's include

Basic maps

1. $z \mapsto rz \quad (r>0)$ "stretch"
2. $z \mapsto e^{i\theta} z$ "rotation"
3. $z \mapsto z+b \quad (b \in \mathbb{C})$ "translation"
4. $z \mapsto 1/z$ "inversion"

Fact: Basic maps 1–4 generate LFT's.

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

$$\begin{array}{r l} & \quad a/c \\ cz+d \,\big) & az + b \\ & az + ad/c \\ \hline & \quad (b - ad/c) \end{array}$$

← remainder

$$L(z) = \frac{a}{c} + \frac{b-ad/c}{cz+d}$$

$c = re^{i\theta}$

$b - \frac{ad}{c} = Re^{i\psi}$

$z \mapsto$ stretch by r, rotate by θ, translate by d, invert, stretch by R, rotate by φ, translate by a/c.

Hmmm. Needed $c \neq 0$. If $c = 0$:

$$L(z) = \frac{az+b}{d} \qquad ad \neq 0$$

so $a \neq 0$, $d \neq 0$.

$$= Az + B \qquad A = \frac{a}{d}, \; B = \frac{b}{d}$$

$$A = re^{i\theta}$$

$L(z) =$ stretch by r, rotate by θ, translate by B.

Say "Basic maps 1-4 generate the LFTs."

Note Basic maps 1-3: $\mathbb{C} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C}$

$$\lim_{z \to \infty} L(z) = \infty \quad \text{for } 1\text{-}3.$$

so $\hat{\mathbb{C}} \xrightarrow[\text{onto}]{1\text{-}1} \hat{\mathbb{C}}$, $\infty \mapsto \infty$

Basic map #4: $\hat{\mathbb{C}} \xrightarrow[\text{onto}]{1\text{-}1} \hat{\mathbb{C}}$ but $0 \mapsto \infty$, $\infty \mapsto 0$

Fact: LFT's are 1-1 onto self maps of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Case $c=0$: $L(z) = Az + B \qquad \infty \mapsto \infty$

Case $c \neq 0$ $\quad L(z) = \dfrac{a + \frac{b}{z}}{c + \frac{d}{z}} \longrightarrow \dfrac{a}{c}$ as $z \to \infty$.

$$\boxed{L(\infty) = \frac{a}{c}}$$

L blows up when $cz + d = 0$

$z = -d/c$

$$\boxed{L\left(-\frac{d}{c}\right) = \infty}$$

Inverse of an LFT: $\quad w = \dfrac{az+b}{cz+d}$

$$czw + dw = az + b$$

$$z(cw - a) = -dw + b$$

$$z = \frac{-dw + b}{cw - a} \cdot \frac{(-1)}{(-1)} = \frac{dw - b}{-cw + a}$$

LFT with same det!

Aha!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\text{Det}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Fact: $\quad L(z) = \dfrac{az+b}{cz+d} \quad \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$L^{-1}(w) = \frac{dw - b}{-cw + aw} \quad \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

Cool fact: $L_1(L_2(z)) = \dfrac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1}$

$$= \frac{Az + B}{Cz + D}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

Map $L \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbb{A}$ "Group homomorphism"

2x2 matrices with non-zero det.

Fact LFT's form a group.

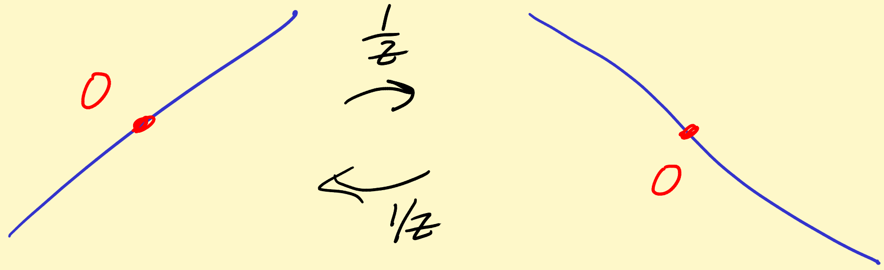
$$id = L(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z$$

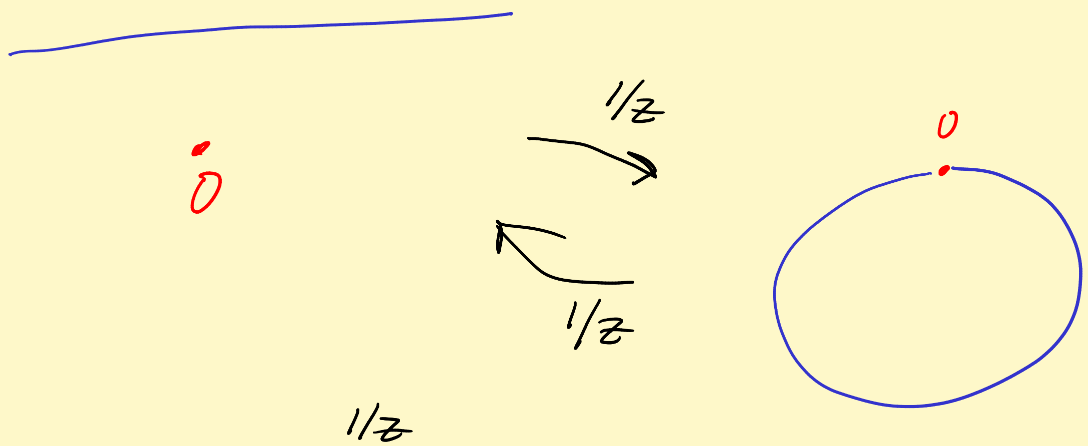
Geometric facts: LFTs map $\left\{\begin{matrix}\text{lines}\\ \text{circles}\end{matrix}\right\}$

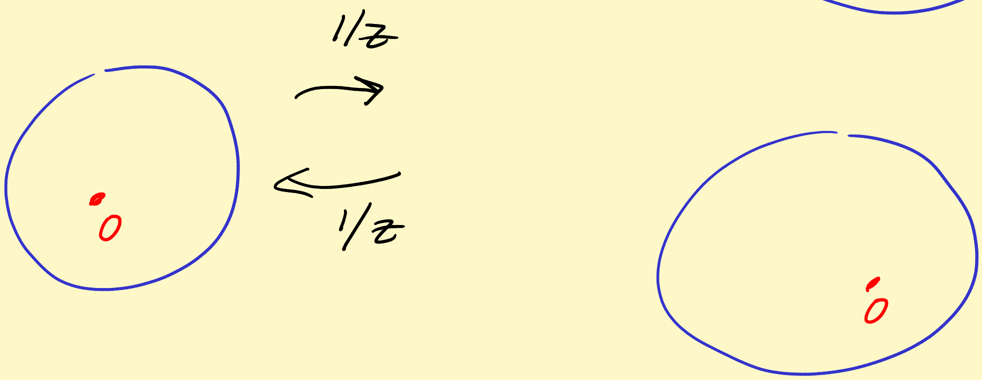
Pf: Basic maps 1-3 do that: In fact

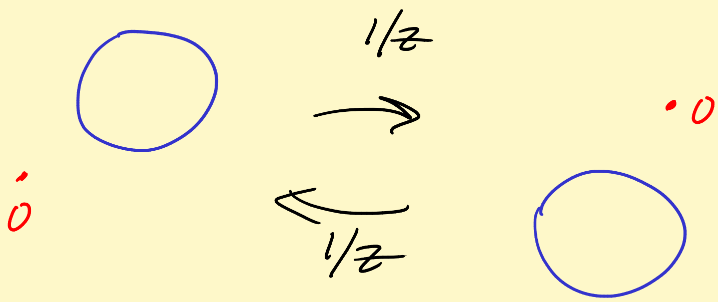
$$\{\text{lines}\} \to \{\text{lines}\}$$
$$\{\text{circles}\} \to \{\text{circles}\}$$

Basic Map 4 jumbles up lines, circles:

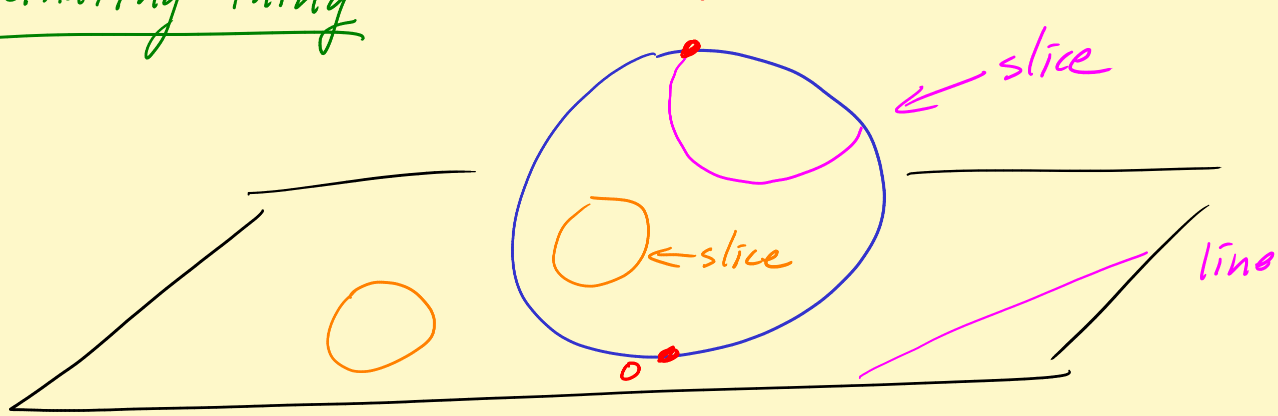








Fascinating thing



{lines, circles} in $\mathbb{C}$ correspond to slices of R. sphere.

<u>Big fact</u> A LFT that fixes 3 pts must be identity.

<u>Why</u>: $L(z_j) = z_j \qquad j = 1, 2, 3$

$$z_j = \frac{a z_j + b}{c z_j + d}$$

$$c z_j^2 + (a-d) z_j - b = 0$$

Hmmm. Quadratic eqn with 3 roots!

Must be that $c = 0$, $\underline{a-d=0}$, $b = 0$.

$$L(z) = \frac{az}{d}, \quad ad \neq 0 \qquad \begin{array}{l} a \neq 0 \\ d \neq 0. \end{array}$$

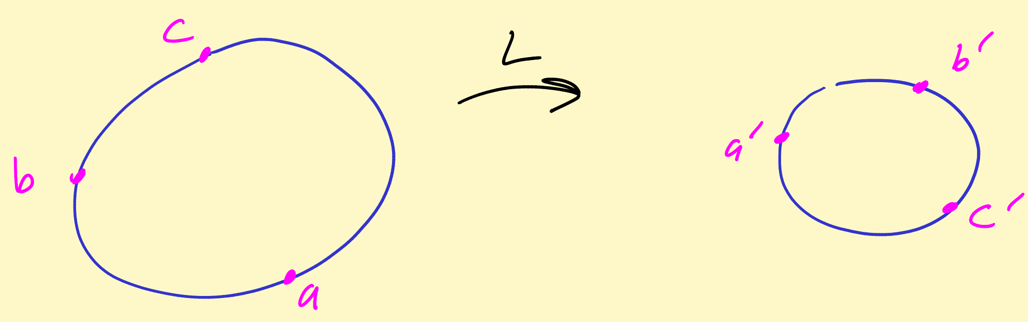
$a - d = 0$ so $a = d$

$$L(z) = z \quad \checkmark$$

<u>Consequence</u> $L_1 = L_2$ at 3 pts $\Rightarrow L_1 \equiv L_2$.

3 points determine a circle!

2 points (plus ∞) determine a line!

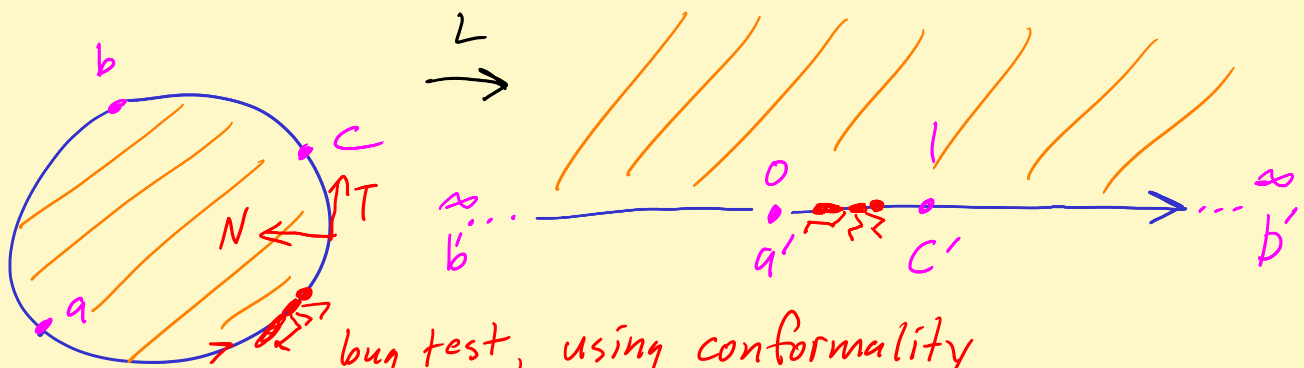


How to cook up LFTs

EX: $L(z) = \dfrac{z-a}{z-b}$ $\qquad a \mapsto 0$, $\quad b \mapsto \infty$

Hmmm: What if I wanted $L(c)=1$ too?

Aha! $\boxed{L(z) = \dfrac{c-b}{c-a} \cdot \dfrac{z-a}{z-b}}$ $\qquad a \mapsto 0$, $\quad b \mapsto \infty$, $\quad c \mapsto 1$



bug test, using conformality shows inside ◯ → UHP.