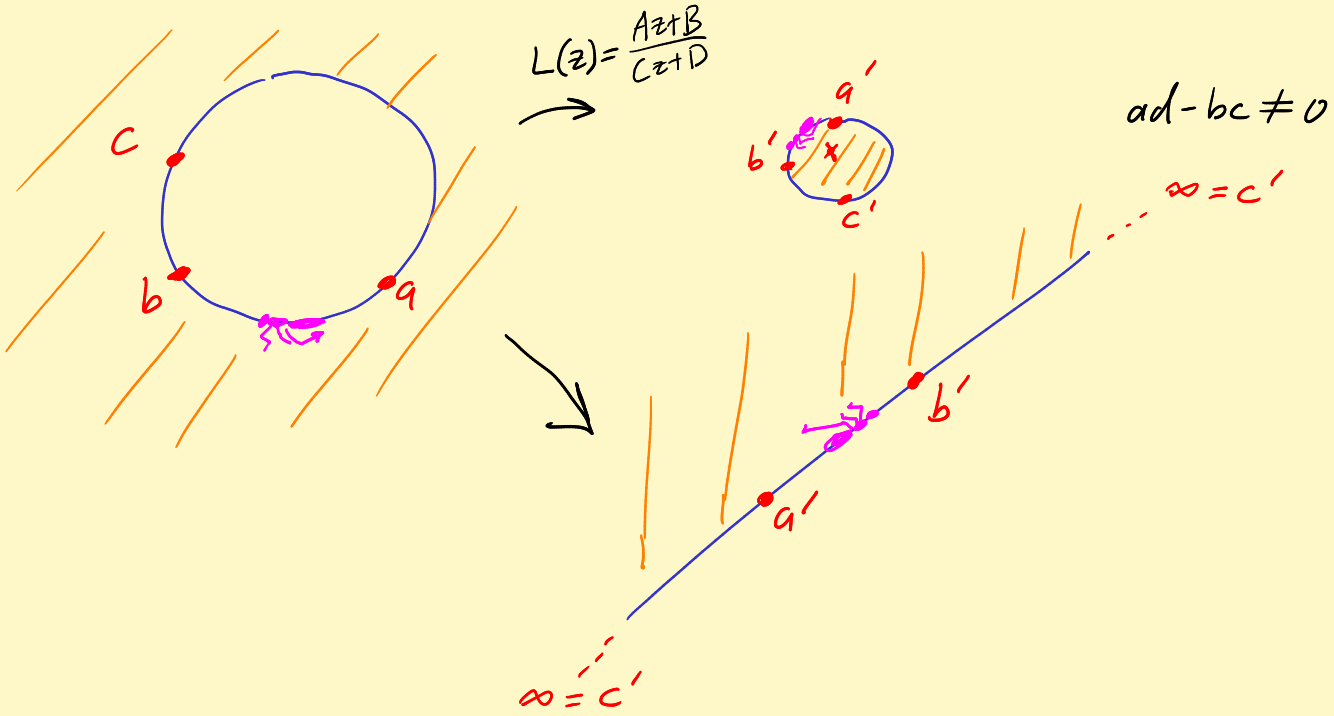
Lesson 38 Applications of linear fractional transformations

Exam 2 due FRIDAY by 11:00 pm in Gradescope

3 pts in $\mathbb{C}$ determine a circle

2 pts in $\mathbb{C}$ (plus the pt at ∞) determine a line



How to cook up LFTs

$$L(z) = \frac{c-b}{c-a} \cdot \frac{z-a}{z-b}$$

$$L: \quad a \longmapsto 0, \quad b \longmapsto \infty, \quad c \longmapsto 1$$

How to find LFT

$$z_1 \longmapsto w_1, \quad z_2 \longmapsto w_2, \quad z_3 \longmapsto w_3$$

z's and w's in $\mathbb{C}$ and distinct

Aha! $L_1(z) = \dfrac{z_3-z_2}{z_3-z_1} \cdot \dfrac{z-z_1}{z-z_2}$

$$L_2(w) = \frac{w_3-w_2}{w_3-w_1} \cdot \frac{w-w_1}{w-w_2}$$

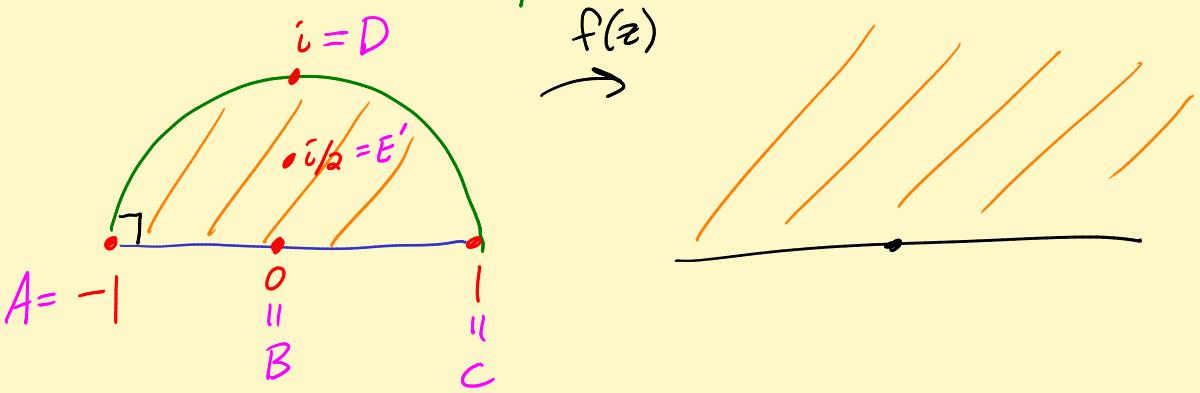
Ans: $L_2^{-1} \circ L_1$

$$L_1 \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$L_2 \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad L_2^{-1} \sim \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

$$L_2^{-1} \circ L_1 \sim \text{Matrix multiplication}$$

Prob Find a conformal map



Idea: Get LFT L that blows a corner out to ∞.

Then circle and line **both** get mapped to lines

Easy: $L(z) = \dfrac{z+1}{z-1}$

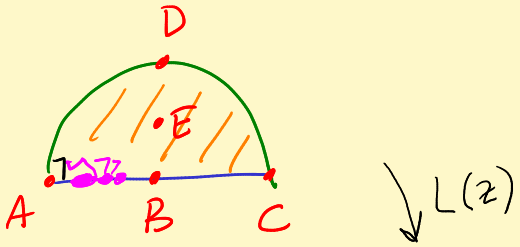
$L(1) = \infty = C'$

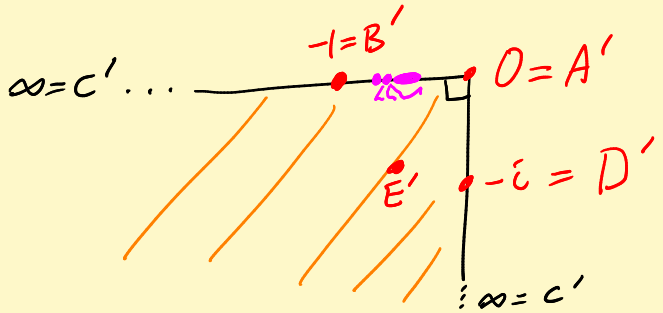
$L(-1) = 0 = A'$

$L(i) = \dfrac{i+1}{i-1} = -i = D'$

$L(0) = \dfrac{0+1}{0-1} = -1 = B'$

$L(\frac{i}{2}) = E'$ ← better to use a bug test!

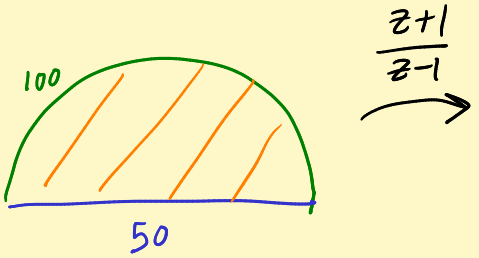




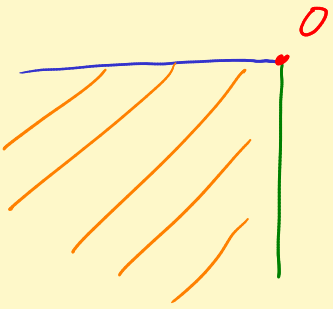
("One pt test" could have shown  →

Conformality near A : Right angle preserved !

Heat problem

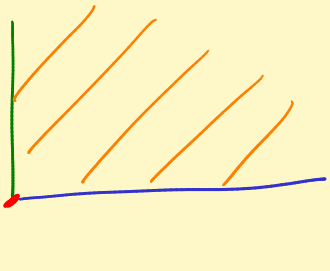


$\frac{z+1}{z-1}$

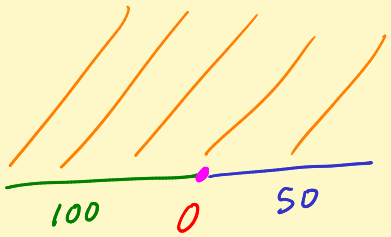


$-z$

$e^{i\pi}z$



z^2



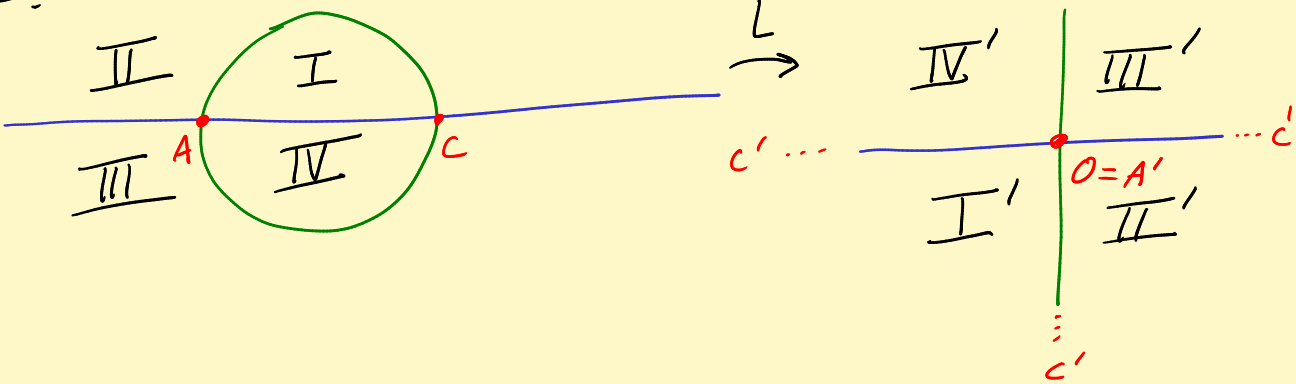
$$f(z) = \left(\frac{z+1}{z-1}\right)^2$$

$$T(z) = \frac{150}{\pi}\operatorname{Arg} z - 50$$

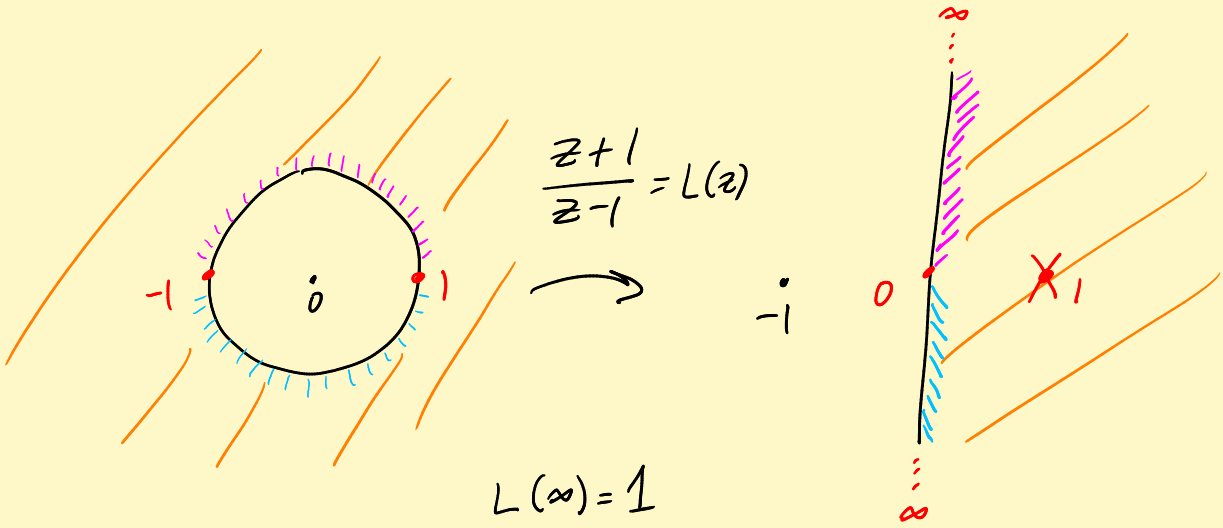
Solution to heat problem on half disc

$$T(f(z)) = \frac{150}{\pi}\operatorname{Tan}^{-1}\left(\frac{\operatorname{Im}\left[\frac{(x+iy)+1}{(x+iy)-1}\right]^2}{\operatorname{Re}\left[\frac{(x+iy)+1}{(x+iy)-1}\right]^2}\right) - 50$$

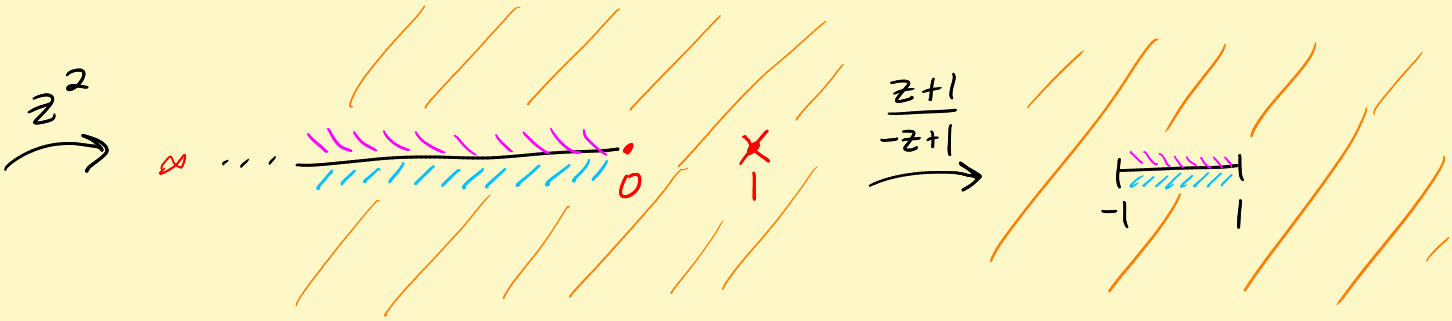
Big picture:



How to cook up Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

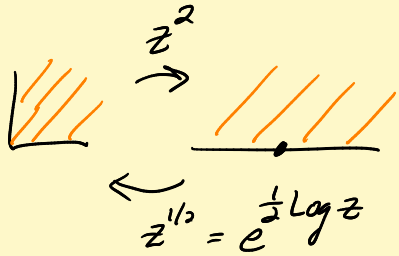


1 gets missed!

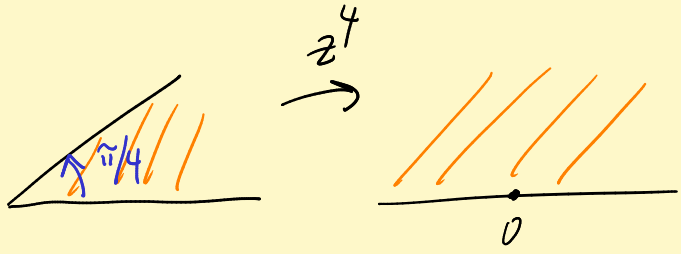


Composition is $J(z)$!

Conformal mapping toolbox 1)

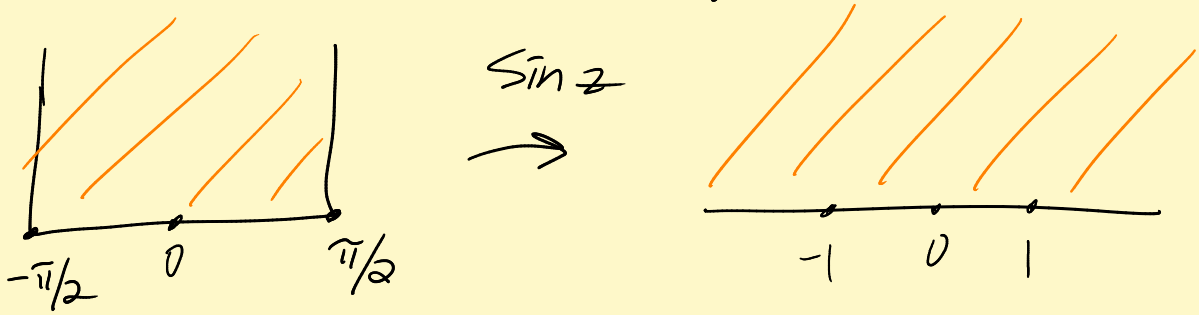


2)

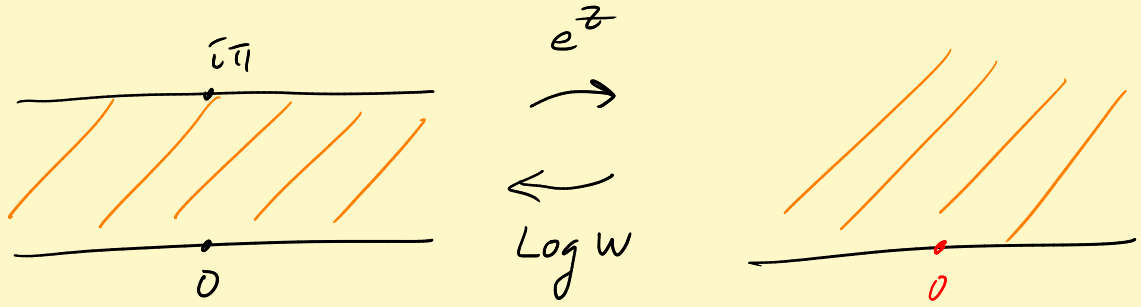


3) $z^\alpha = e^{\alpha \text{Log}\, z}$ opens angles by α factor.

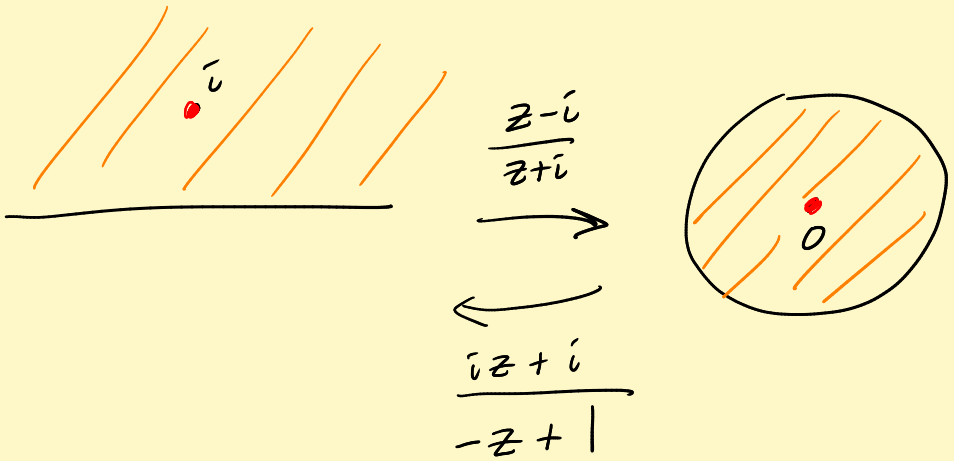
4)



5)



6)



Heat problems: use conformal maps plus facts:

Re (analytic) = harmonic ← two way fact

Composition of analytic = analytic