Lesson 39 Applications of conformal mapping

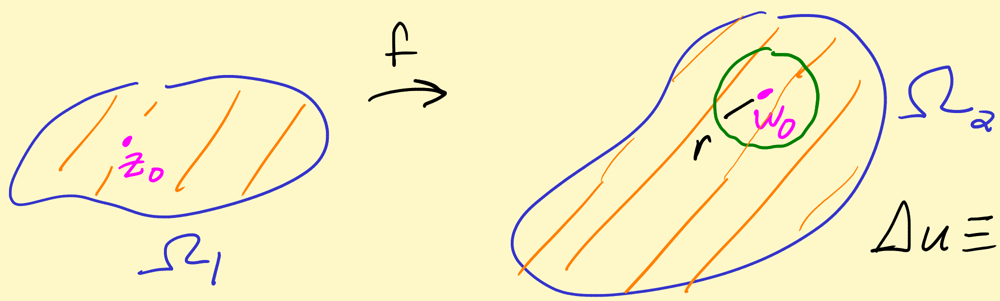
Exam 2 due in GS by 11:00pm tonight

Ω_1, Ω_2 domains, $f: \Omega_1 \to \Omega_2$ analytic. If u is harmonic on Ω_2, then $u \circ f$ is harmonic.

Pf: Cauchy-Riemann eqns, $\mathbb{R}^2 \to \mathbb{R}^2$ chain rule,

$\Delta(u \circ f) = 0$. Mess!

$f(z_0) = w_0$



$\Delta u \equiv 0$

u has a harm conj on $D_r(w_0) \subset \Omega_2$.

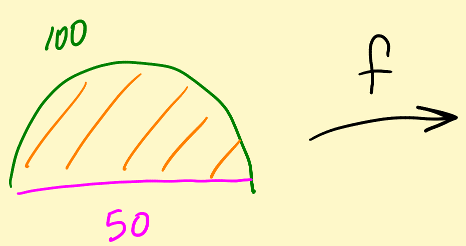
Write $F = u + iv$ on $D_r(w_0)$, analytic on $D_r(w_0)$.

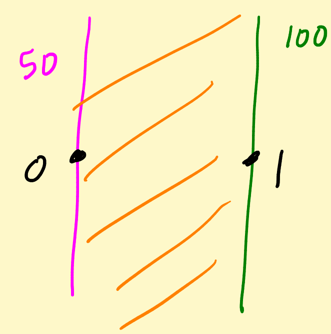
f cont at z_0. So $\exists\, \delta > 0$ such that $D_\delta(z_0) \subset \Omega_1$ and $|f(z) - f(z_0)| < r$ when $|z - z_0| < \delta$ $\Rightarrow$

$$f(D_\delta(z_0)) \subset D_r(w_0).$$

Now $F \circ f$ is analytic on $D_\delta(z_0)$.

So $u \circ f = \mathrm{Re}(F \circ f)$ is harmonic on $D_\delta(z_0)$.

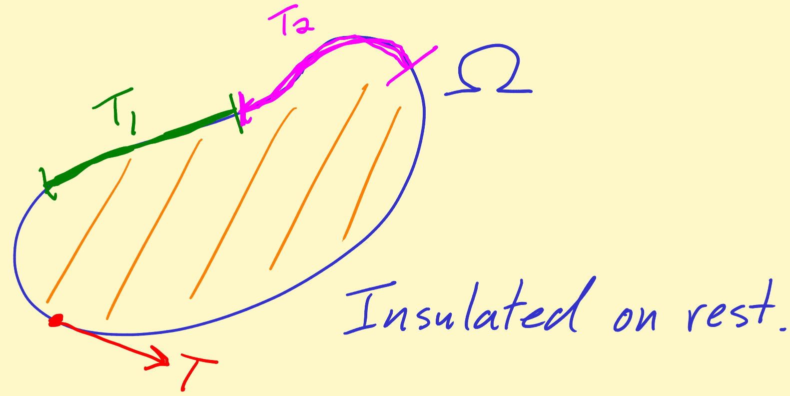




$u = 50(\mathrm{Re}\, z) + 50$

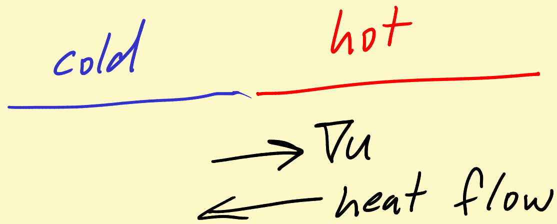
$u \circ f$ solves prob here.

Fancier heat problem:



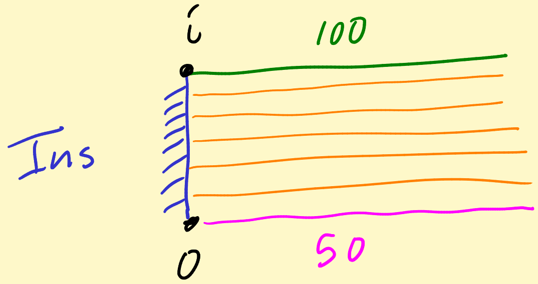
Insulated: No heat flows in or out

Kelvin's Law: Rate and direction of heat flow is proportional to <u>minus</u> temp gradient

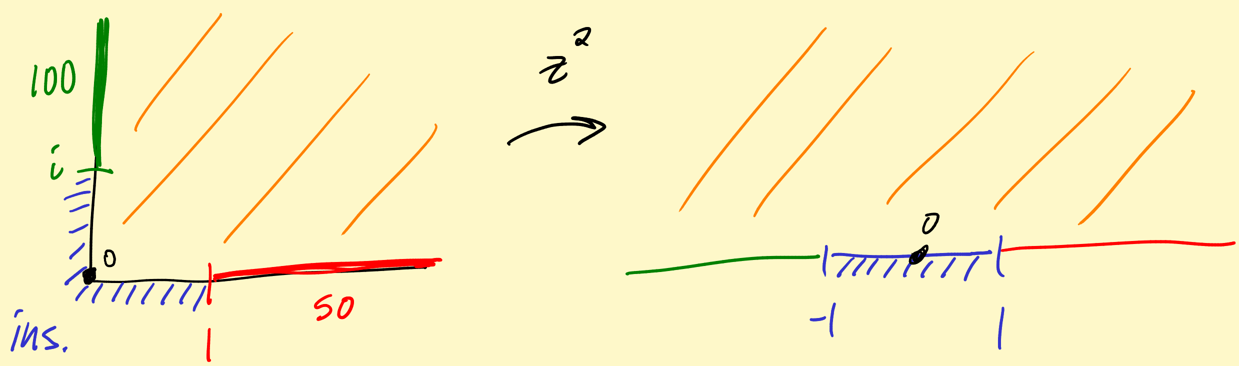


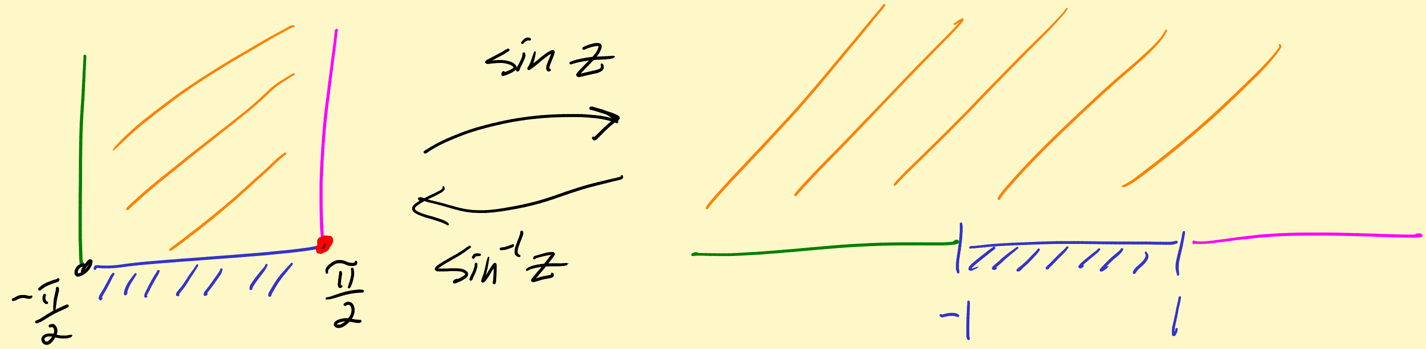
Insulated on bndry: $\nabla u \perp$ (normal vector)

Easy problem:

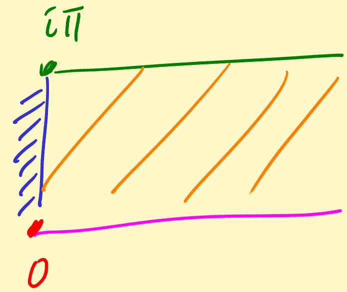


$u = 50(\operatorname{Im} z) + 50$



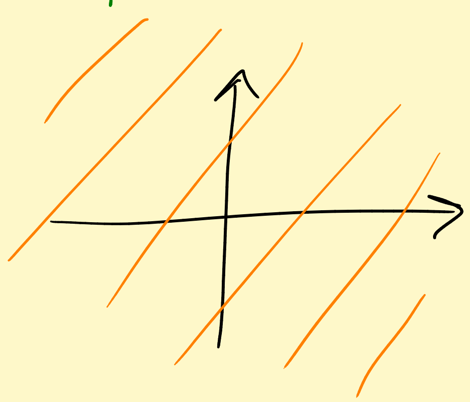


$$\underbrace{e^{-i\frac{\pi}{2}}}_{=-i}\left(z-\frac{\pi}{2}\right) \longrightarrow$$

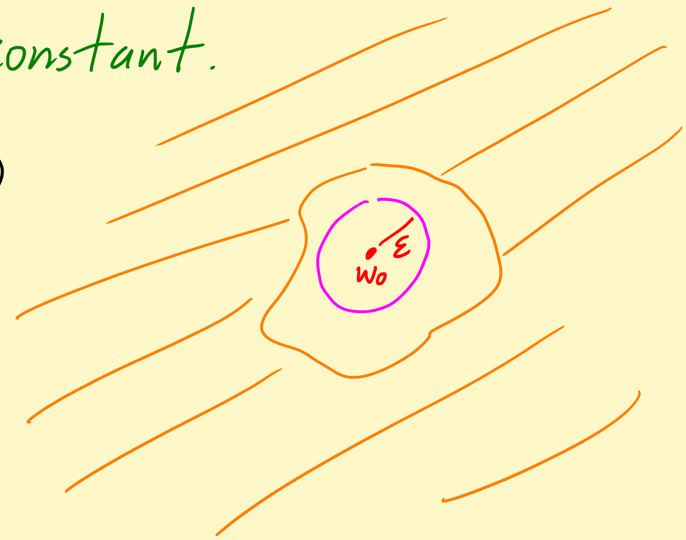


<u>Casorati–Weierstraß theorem</u> An entire fcn that misses an open set must be constant.

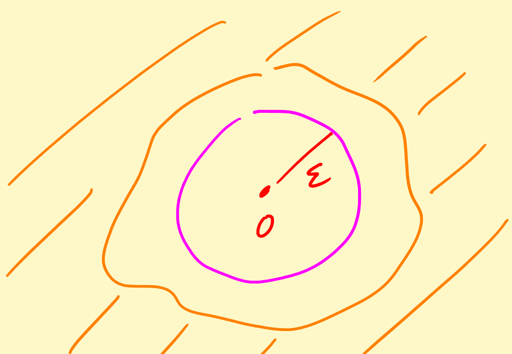
Pf



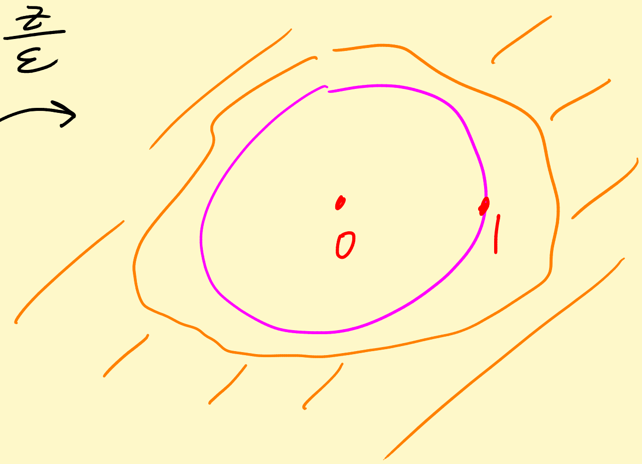
$\xrightarrow{f(z)}$



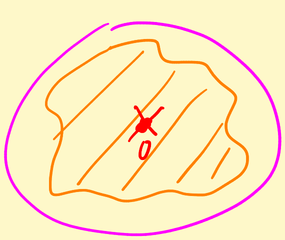
$\xrightarrow{z-w_0}$



$\xrightarrow{\frac{z}{\varepsilon}}$



$\xrightarrow{1/z}$



Aha! Bounded entire fcn!

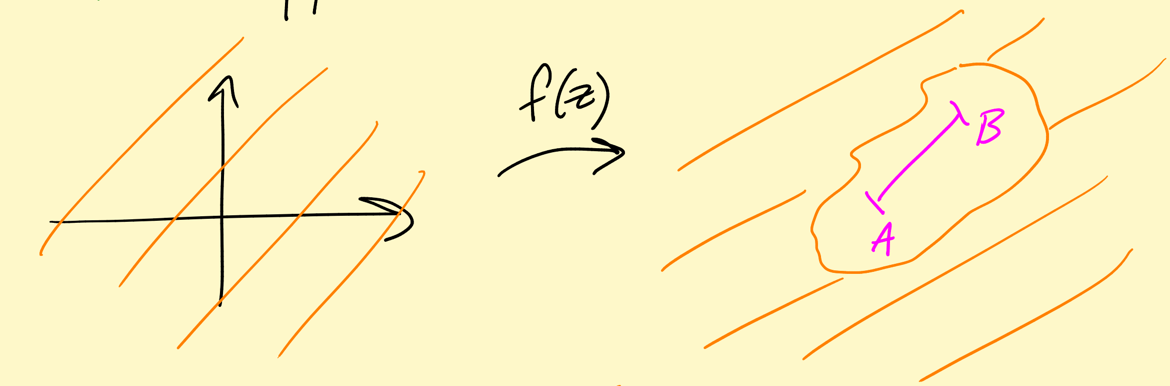
$\dfrac{\varepsilon}{f(z)-w_0}$ is bndd entire. Liouville's $\Rightarrow$

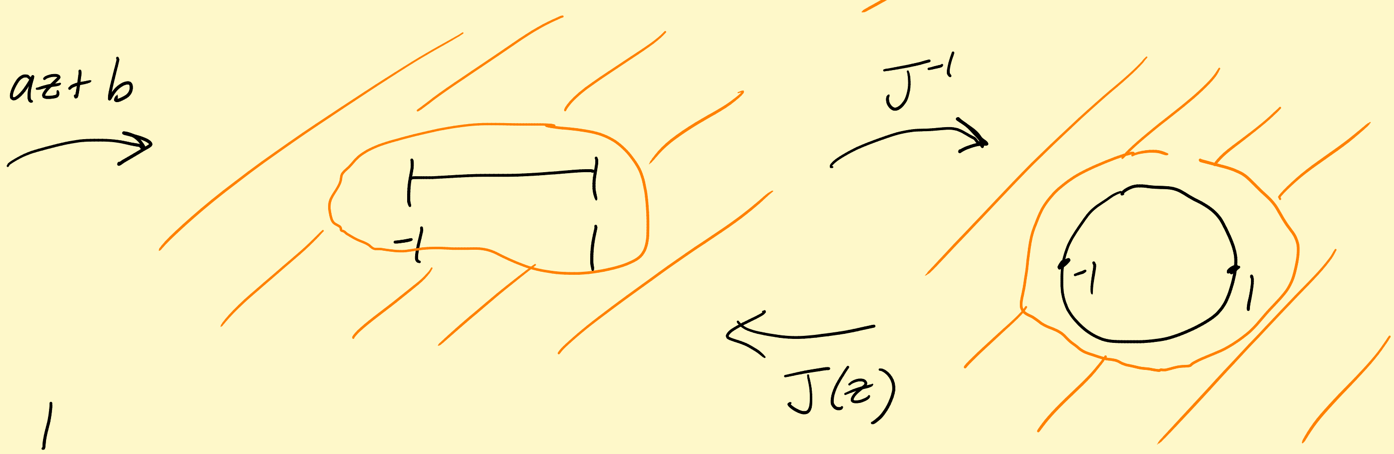
it's constant. 0 not hit. So

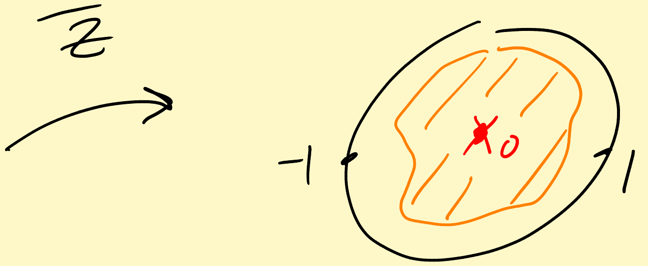
$$\frac{\varepsilon}{f(z)-w_0} = c \neq 0 .$$ Solve for $f(z)$.

See $f(z) = \text{const.}$

Hmmm. Suppose f entire and misses a line segment.







bndd entire fcn!

An entire fcn that misses a line segment must be const.

Picard's little theorem An entire fcn that misses two points in $\mathbb{C}$ must be constant.

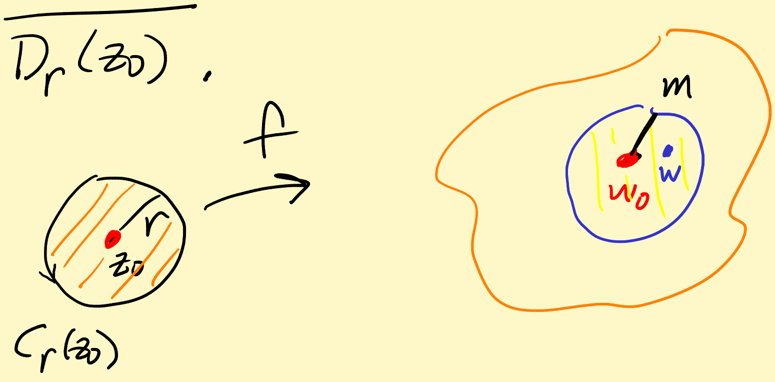
[Note: e^z misses 0, so one is ok to miss.
(Sin z doesn't miss any.)]

Rouché's theorem $\Longrightarrow$ Open mapping theorem:
Non-constant analytic fcns map open sets to open sets.

f analytic. $f(z_0) = w_0$. f not constant.

Then $f(z) - w_0$ has an isolated zero at z_0.

So $\exists\, r > 0$ such z_0 is only zero of $f(z) - w_0$ on $\overline{D_r(z_0)}$.



Want to see that w gets hit.

Let $m = \min\limits_{z \in C_r(z_0)} |f(z) - w_0| > 0$

Aha! Want $f(z) - w$ and $f(z) - w_0$ to have same # zeroes

<u>Rouché</u>:

$|H(z)| < |F(z)|$ on $C_r(z_0)$

$\underbrace{F}_{f-w_0}$ and $\underbrace{F+H}_{f-w}$ have same # zeroes inside $C_r(z_0)$.

$H = (F+H) - F = w_0 - w$ Aha!

$|H| = |w_0 - w| \overset{\text{want}}{<} m \leq |f(z) - w_0|$ on $C_r(z_0)$

Rouché's $\Rightarrow$ $D_m(w_0) \subset f(D_r(z_0))$ ✓