Lesson 40 Schwarz lemma and consequences

Review on Wed, Fri.
review problems on home page

Schwarz lemma Suppose f is analytic on $D_1(0)$ and $f: D_1(0) \circlearrowleft$ $[\,|f(z)|<1$ when $|z|<1\,]$, and $f(0)=0$. Then

A) $|f(z)| \le |z|$ for $z \in D_1(0)$, and

B) $|f'(0)| \le 1$.

Furthermore, if equality holds in (A) for some $z \ne 0$, or equality holds in (B), then $f(z) \equiv \lambda z$ for some constant λ with $|\lambda| = 1$. [Think $\lambda = e^{i\alpha}$ and $\lambda z = e^{i\alpha} z$ is a rotation.]

Pf: Define

$$F(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \ne 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

$$DQ = \frac{f(z) - \underbrace{f(0)}_{0}}{z - 0} = \frac{f(z)}{z} \longrightarrow f'(0) \quad \text{as} \quad z \to 0$$

F is continuous at $z=0$. Riemann removable sing. thm $\Rightarrow$ F is analytic on $D_1(0)$.

Claim: $|F(z)| \le 1$ on $D_1(0)$.

Hmmm. If $|z| = r$, $0 < r < 1$ $\quad |F(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{r} < \frac{1}{r}$

Big idea! Let $r \nearrow 1$.

Pick $z_0 \in D_1(0)$. Choose r with $|z_0| < r < 1$.

Max princ: $\quad |F(z_0)| \le \underset{\overline{D_r(z_0)}}{\text{Max}} |F| = \underset{C_r(z_0)}{\text{Max}} |F| < \frac{1}{r}$

Aha! Let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$ and we see that $|F(z_0)| \le 1$. Since z_0 arbitrary, get

$$\underbrace{|F(z)| \le 1}\ \text{on}\ D_1(0).$$

$\left|\frac{f(z)}{z}\right| \le 1 \quad z \ne 0 \quad$ (A) ✓ $\quad |\underbrace{f(0)}_{=0}| \le |0|$ ✓

$|f'(0)| \le 1 \quad z = 0 \quad$ (B) ✓

Part 2: If equality held at $z \ne 0$, or $z = 0$, then $|F(z)|$ would attain max value of 1 at $z \in D_1(0)$.

$\Longrightarrow$ $F(z) \equiv$ const., which must have modulus 1.

Big consequence The one-to-one analytic maps of $D_1(0)$ of onto $D_1(0)$ are $\quad e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z}$, $\quad a \in D_1(0)$.

$\longleftarrow$ Möbius transformations.

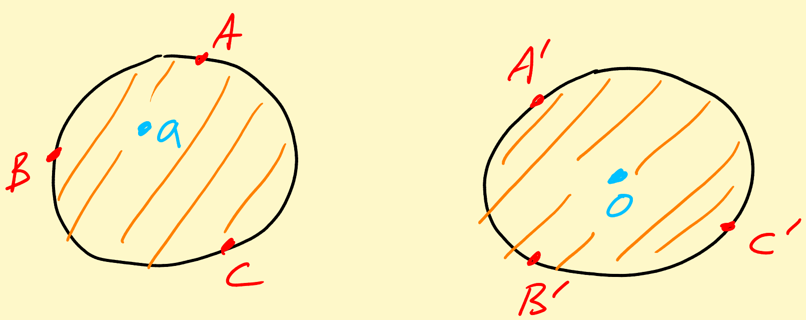
Why: Step 1. $a \in D_1(0)$. $\dfrac{z-a}{1-\bar{a}z} : D_1(0) \circlearrowleft$ 1-1, onto

$$\frac{e^{it}-a}{1-\bar{a}e^{it}} \cdot \frac{e^{-it}}{e^{-it}} = \frac{e^{-it}\overbrace{\left(e^{it}-a\right)}^{w}}{\underbrace{\left(e^{-it}-\bar{a}\right)}_{\bar{w}}}$$

$$= \underbrace{e^{-it} \cdot \frac{w}{\bar{w}}}_{\text{unimodular.}}$$

Unit circle $\longrightarrow$ unit circle.

One pt test for LFTs: $a \longmapsto 0$



$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

$$= \frac{z-a}{-\bar{a}z+1}$$

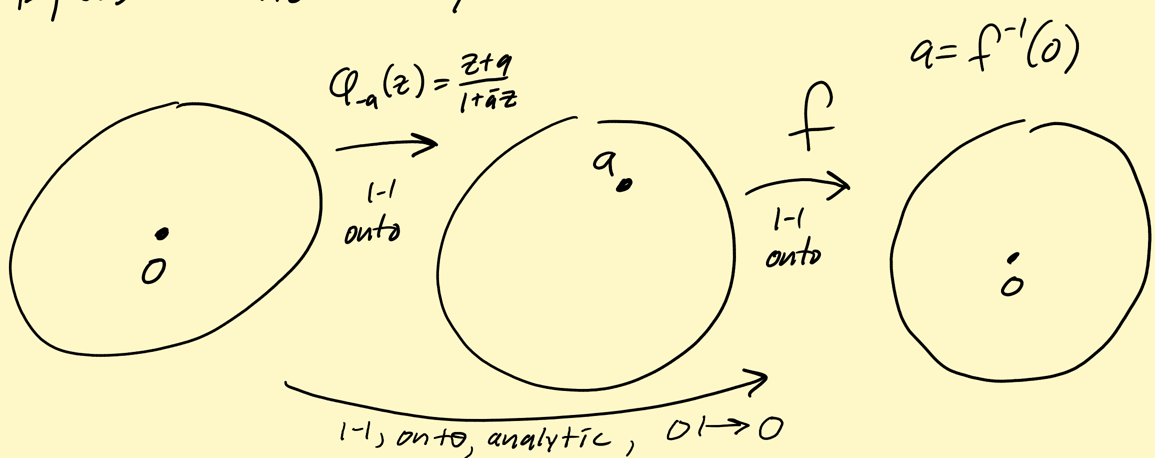
$$\varphi_a^{-1}(z) = \frac{1\cdot z+a}{\bar{a}z+1}$$

$$= \frac{z+a}{1+\bar{a}z}$$

Suppose $f : D_1(0) \circlearrowleft$ 1-1 onto analytic.

Hmmm.

Say $a = f^{-1}(0)$



$$F(z) = (f \circ \varphi_{-a})(z).$$

$\boxed{F(0) = 0}$ Schwarz $\Rightarrow$ $\underbrace{|F'(0)| \leq 1}$.

F^{-1} is also 1-1, onto analytic : $D_1(0)$ ↺ analytic

$$F^{-1}(0) = 0 \text{ too!}$$

Schwarz : $\underbrace{|(F^{-1})'(0)|}_{\frac{1}{|F'(0)|}} \leq 1$

$|F'(0)| \geq 1$ too!

Conclude $|F'(0)| = 1$. Schwarz: $F(z) = \lambda z$
$|\lambda| = 1$.

Almost done. $f(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda z$

$z = \frac{w-a}{1-\bar{a}w}$

$w = \varphi_{-a}(z)$

$z = \varphi_{-a}^{-1}(w)$

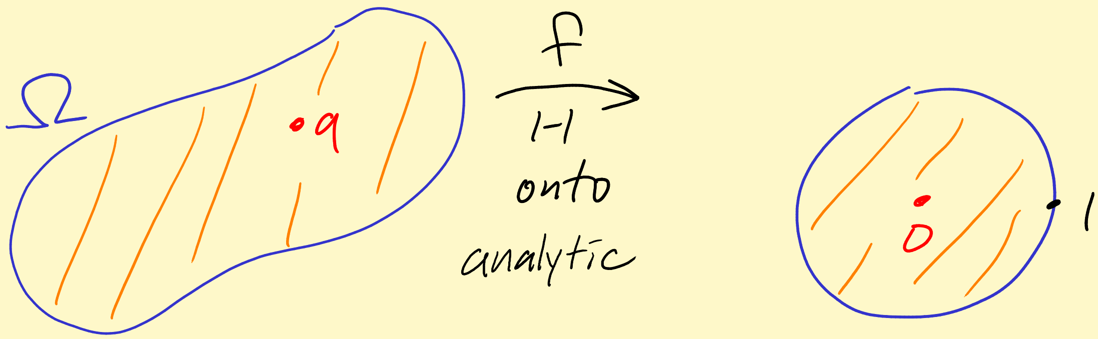
$z = \varphi_a(w) = \frac{w-a}{1-\bar{a}w}$

So $f(w) = \lambda \frac{w-a}{1-\bar{a}w}$ ✓

Automorphism group of $D_1(0)$ = $Aut(D_1(0))$

= grp of all 1-1 onto conformal self maps of $D_1(0)$.

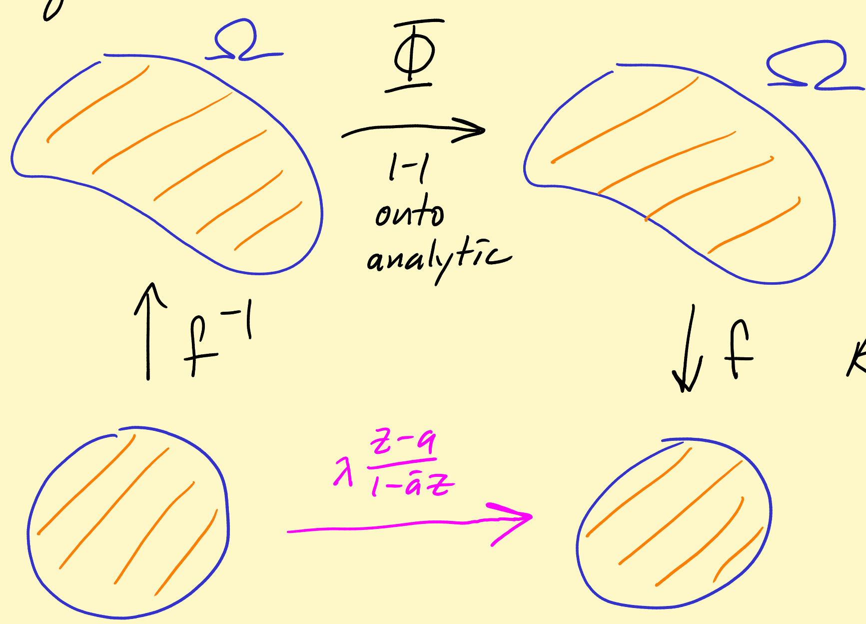
<u>Important fact</u> Schwarz lemma is a key ingredient of pf of the Riemann mapping theorem.



f is a Riemann map

<u>Fact</u>: Schwarz $\Rightarrow$ $g : \Omega \to D_1(0)$ analytic, and $g(a)=0$, then $|g'(a)| \le |f'(a)|$.

The Riemann map maximizes $|g'(a)|$ among all analytic $g : \Omega \to D_1(0)$. That's how we cook it up! up.



Riemann map

See $Aut(\Omega)$ is easily gotten from $Aut(D_1(0))$ via conjugation with a Riemann map! [MA 530]