Practice problems review 1

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2. Suppose that f and g are analytic on a disc about a and that each has a zero of multiplicity N at a. Prove a L'Hôpital's formula: $\lim_{z\to a}\frac{f(z)}{g(z)} = \frac{f^{(N)}(a)}{g^{(N)}(a)}$.

$$f(z) = \underset{\overset{\|}{0}}{a_0} + \underset{\overset{\|}{0}}{a_1}(z-a) + \cdots + \underset{\overset{\|}{0}}{a_{N-1}}(z-a)^{N-1} + a_N(z-a)^N + \cdots$$

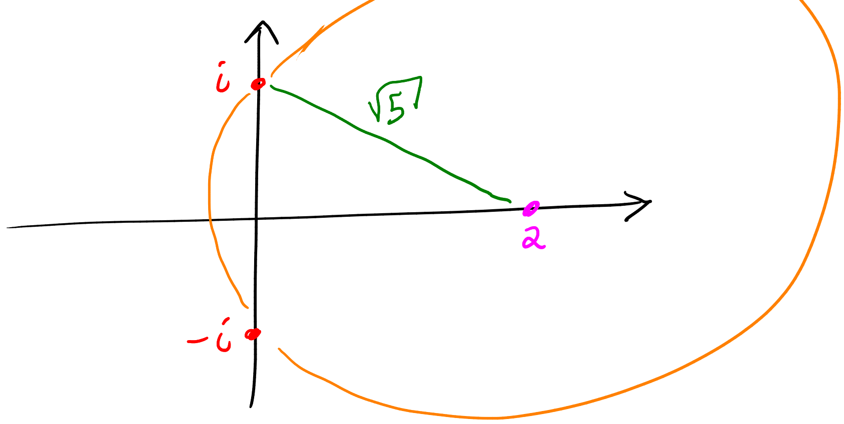
$$a_N = \frac{f^{(N)}(a)}{N!}$$

$$= (z-a)^N F(z) \quad \text{where } F(z) = a_N + a_{N+1}(z-a) + \cdots$$

$$F(a) = a_N$$

$$\frac{f(z)}{g(z)} = \frac{(z-a)^N F(z)}{(z-a)^N G(z)} \to \frac{\left(\frac{f^{(N)}(a)}{N!}\right)}{\left(\frac{g^{(N)}(a)}{N!}\right)} \checkmark$$

3. Let $f(z) = \frac{1}{1+z^2}$. Find the first few terms in the power series $\sum_{n=0}^{\infty} a_n(z-2)^n$ for f centered at $z = 2$. What is the radius of convergence of the series? Explain.



$R \geq \sqrt{5}$

$f(z)$ blows up at $\pm i$, so $R \leq \sqrt{5}$ too.

So $R = \sqrt{5}$.

Could use Taylor's to get some terms.

How to get them all: $\frac{1}{1+z^2} = \frac{A}{z-i} + \frac{B}{z+i}$

$$\frac{A}{z-i} = \frac{A}{z-2+2-i} \cdot \frac{\left(\frac{1}{2-i}\right)}{\left(\frac{1}{2-i}\right)}$$

$$\boxed{1 + r + r^2 + \cdots = \frac{1}{1-r} \qquad |r| < 1}$$

$$= \frac{A/(2-i)}{1 - \underbrace{\left[\frac{-(z-2)}{2-i}\right]}_{r}}$$

$$= \frac{A}{(2-i)}\left[1 - \frac{(z-2)}{2-i} + \frac{(z-2)^2}{(2-i)^2} - \cdots \right]$$

etc.

4. What is the radius of convergence of the power series for $\frac{\sin z}{z}$ centered at $z = \pi$?

$\frac{\sin z}{z}$ has an isolated sing at $z=0$.

It's removable. Power series at $z=\pi$

same as power series for

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases},$$

which is entire! So R of C $= \infty$.

$\frac{\operatorname{Log} z}{z-1} \leftarrow z=1$ is removable.

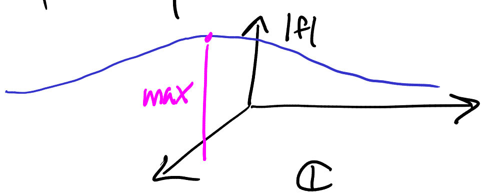
R about 2 is 2, not 1.

Max Princ $\Rightarrow$ Fund Thm Alg.

$P(z)$. Basic poly est $a|z|^N < |P(z)| < A|z|^N$, $|z| > R_0$.

No zeroes. $|f(z)| = \left|\frac{1}{P(z)}\right| \le \frac{1}{a|z|^N}$, $|z| > R_0$

$\to 0$ as $|z| \to \infty$.



$\Rightarrow f \equiv$ const,

const must $=0$. ↯

7. Suppose that f and g are analytic on a disc about a point a and suppose that g has a zero of multiplicty two at a. Show that the residue of f/g at a is equal to

$$\frac{6g''(a)f'(a) - 2g'''(a)f(a)}{3g''(a)^2}.$$

$$\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$$

a simple zero of g.

$$g(z) = a_2(z-a)^2 + a_3(z-a)^3$$

$$= (z-a)^2\left[a_2 + a_3(z-a) + \cdots\right] = (z-a)^2 G(z)$$

$$a_2 = \frac{g''(a)}{2!} \qquad a_3 = \frac{g'''(a)}{3!}$$

$$\begin{cases} G(a) = \dfrac{g''(a)}{2!} \\ G'(a) = \dfrac{g'''(a)}{3!} \end{cases}$$

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)^2 G(z)} = \frac{1}{(z-a)^2}\Bigg[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-a) + \cdots}\Bigg]$$

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{(z-a)} + \cdots$$

Res!

$$A_1 = \frac{1}{1!}\left[\frac{f(z)}{G(z)}\right]'\Bigg|_{z=a} =$$