Practice problems review 2

Final exam Monday, April 29, 7:00–9:00 pm in LWSN B155

(TWO two-sided crib sheets handwritten on both sides)

7. Suppose that f and g are analytic on a disc about a point a and suppose that g has a zero of multiplicty two at a. Show that the residue of f/g at a is equal to

$$\frac{6g''(a)f'(a) - 2g'''(a)f(a)}{3g''(a)^2}.$$

$$\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$$

a simple zero of g.

$$g(z) = \underbrace{a_2}_{\frac{g''(a)}{2!}\neq 0}(z-a)^2 + \underbrace{a_3}_{\frac{g'''(a)}{3!}}(z-a)^3$$

$$= (z-a)^2 \underbrace{\left[a_2 + a_3(z-a) + \cdots\right]}_{G(z)} = (z-a)^2 G(z)$$

$a_2 = \frac{g''(a)}{2!}$ $\qquad$ $a_3 = \frac{g'''(a)}{3!}$

$$\begin{cases} G(a) = \frac{g''(a)}{2!} \\ G'(a) = \frac{g'''(a)}{3!} \end{cases}$$

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)^2 G(z)} = \frac{1}{(z-a)^2}\Bigg[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-a) + \cdots}\Bigg]$$

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{(z-a)} + \cdots$$

Res!

$$A_1 = \frac{1}{1!}\left[\frac{f(z)}{G(z)}\right]'\Bigg|_{z=a} = \frac{f'(a)G(a) - f(a)G'(a)}{G(a)^2}$$

$$= \frac{f'(a)\frac{g''(a)}{2!} - f(a)\frac{g'''(a)}{3!}}{\left[\frac{g''(a)}{2!}\right]^2}$$

✓

8. If the complex numbers a_n tend to zero as n tends to infinity, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is at least one.

Best: $a_n \to 0$ as $n \to \infty$. So $\exists N$ such that $|a_n| < 1$ if $n \geq N$.

So $|a_n z^n| < |z|^n$ when $n \geq N$.

Compare with geom series $\sum_0^\infty z^n$. Conclude series converges inside $\{z : |z| < 1\}$. So $R \geq 1$.

$\underline{\underline{\text{or}}}$ Hadamard: $\frac{1}{R} = \lim_{N \to \infty} \sup_{n \geq N} \sqrt[n]{|a_n|}$

$a_n \to 0$. So $|a_n| < 1$ when $n \geq N_0$.

$\sqrt[n]{|a_n|} < 1$ when $n \geq N_0$

So $\sup_{n \geq N_0} \sqrt[n]{|a_n|} \leq 1$

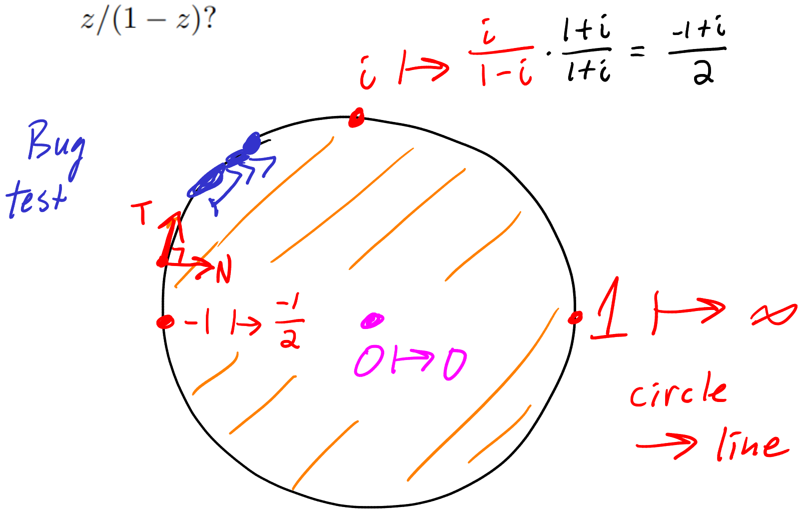
Decreasing seq in N.

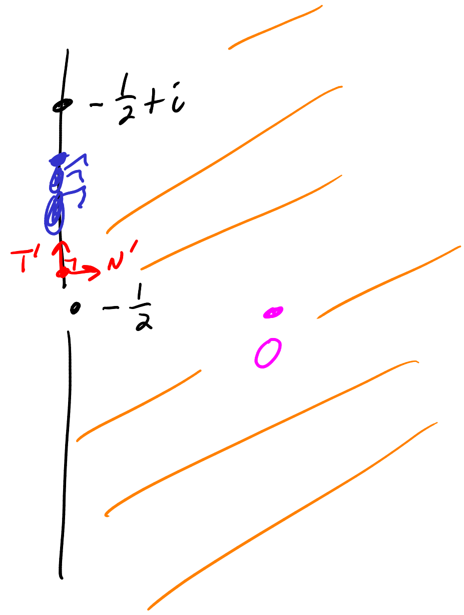
$\lim \sup_{n \geq N} \leq 1$

$\frac{1}{R} \leq 1$

$R \geq 1$

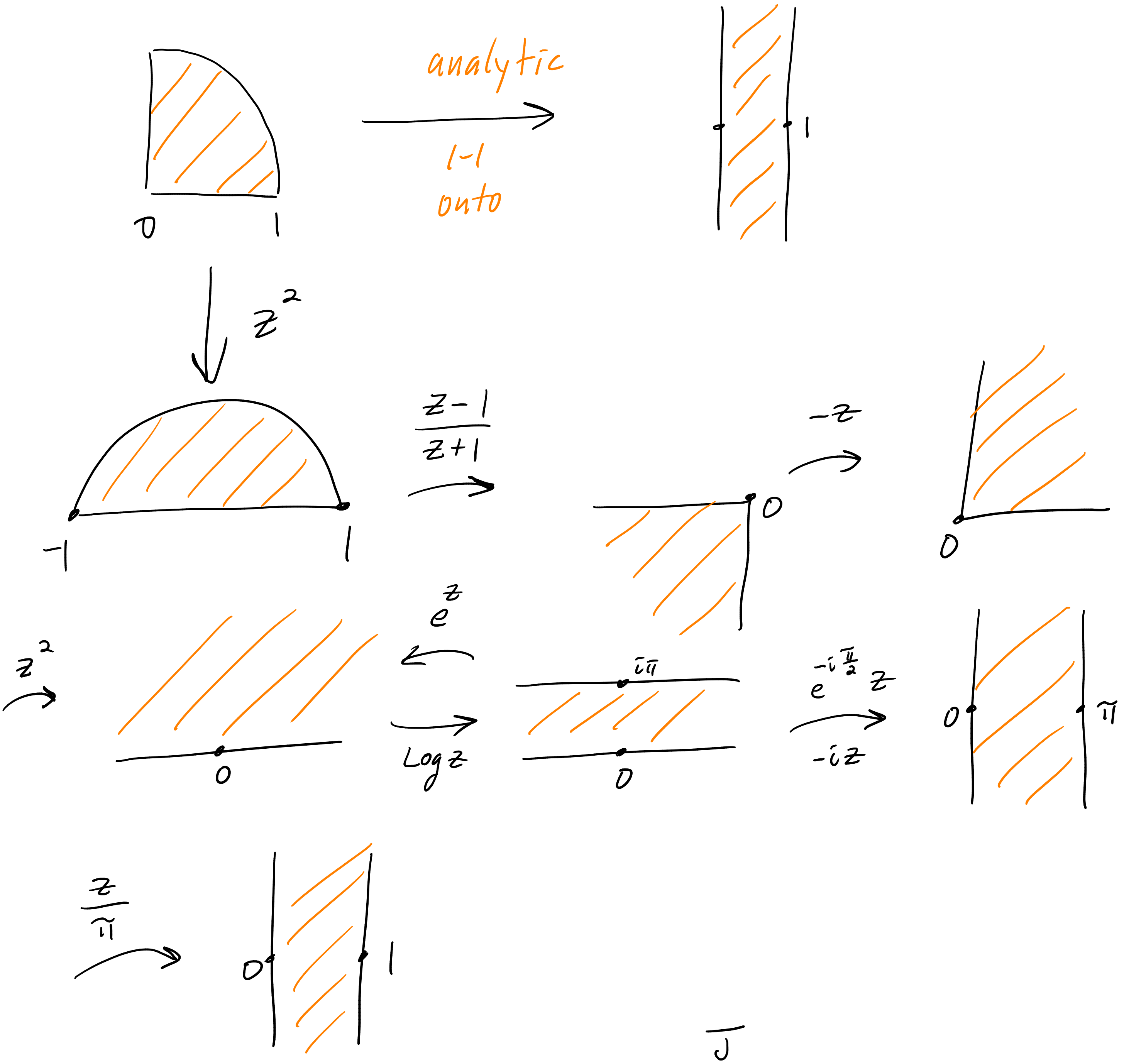
11. What is the image of the unit disc under the linear fractional transformation $T(z) = z/(1-z)$?





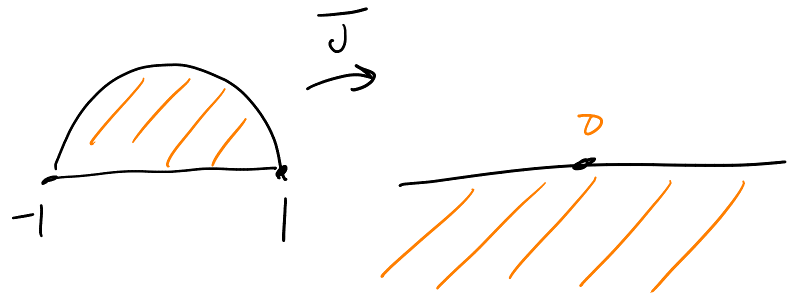
One pt test: $T(0) = 0$

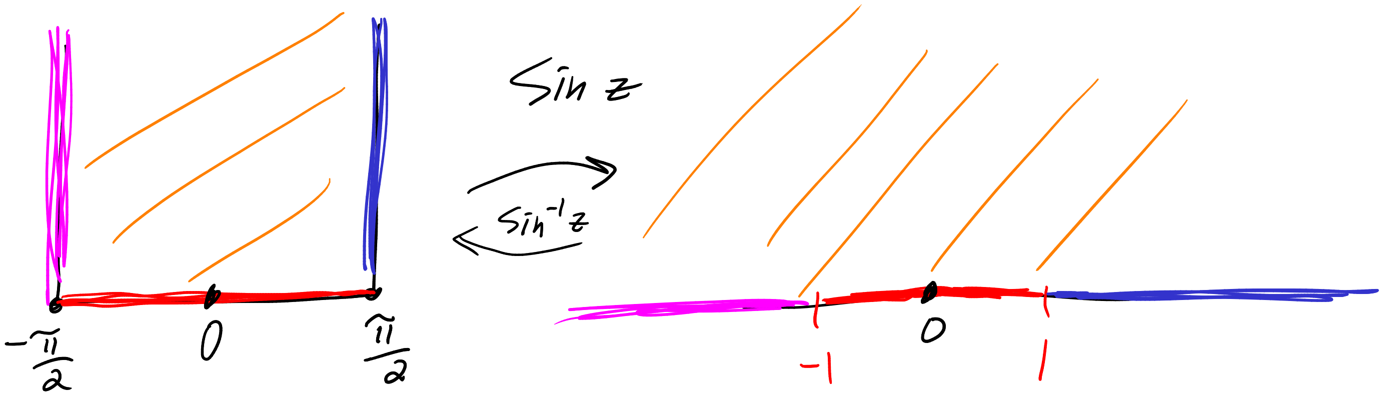
10. Find an analytic function that maps the quarter disc $\{re^{i\theta} : 0 < r < 1,\ \ 0 < \theta < \pi/2\}$ one-to-one onto the strip $\{z : 0 < \text{Re}\, z < 1\}$.



Jukovsky map:

$J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$





$$w = \frac{e^{iz} - e^{-iz}}{2i}$$

$$2iw = e^{iz} - \frac{1}{e^{iz}}$$

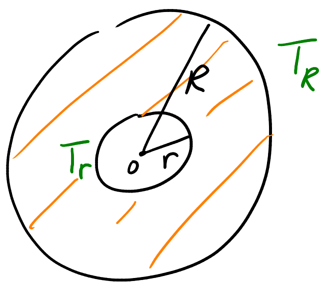
$$0 = \left(e^{iz}\right)^2 - (2iw)\, e^{iz} - 1$$

Quad form: $$e^{iz} = \frac{2iw + \sqrt{-4w^2 + 4}}{2}$$

$$e^{iz} = iw + \sqrt{1 - w^2}$$

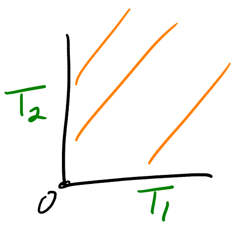
$$z = \frac{1}{i} \log\left(iw + \sqrt{1-w^2}\right)$$

Heat problems from 3.4



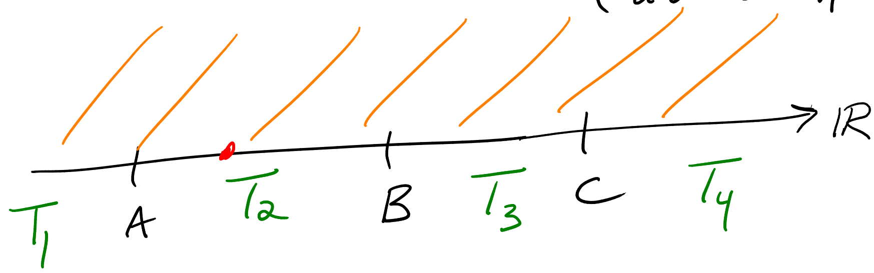
$$u = A \operatorname{Ln}|z| + B$$

$$\begin{cases} A \operatorname{Ln} R + B = T_R \\ A \operatorname{Ln} r + B = T_r \end{cases}$$



$$u = a \operatorname{Arg} z + b$$

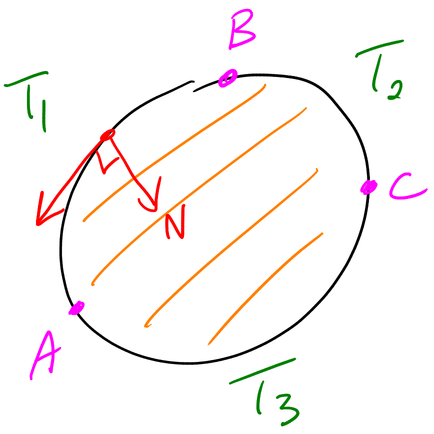
$$\begin{cases} a \frac{\pi}{2} + b = T_2 \\ a \cdot 0 + b = T_1 \end{cases}$$

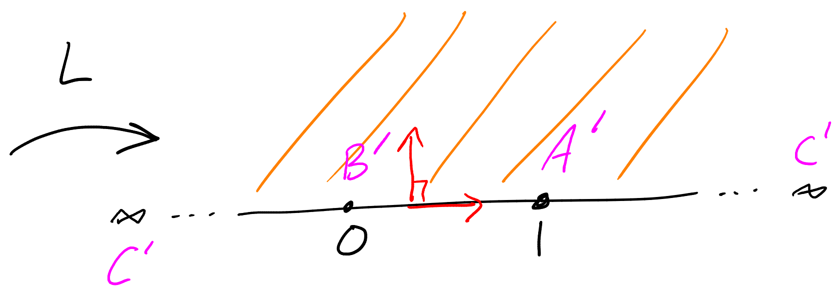


$$u = \underbrace{a \operatorname{Arg}(z-A)}_{} + b \operatorname{Arg}(z-B) + c \operatorname{Arg}(z-c) + d$$

$$\begin{cases} 0 & z = x > A \quad x \text{ to right of } A \text{ on } \mathbb{R} \\ \pi & z = x < A \quad x \text{ to left of } A \end{cases}$$

Can use conformal maps





$$u = a \operatorname{Arg} z + b \operatorname{Arg}(z-1) + c$$

$$L(z) = \frac{A-C}{A-B} \cdot \frac{z-B}{z-C}$$

Sol^n on disc: $u(L(z))$

Exam 2: Ave 84

Off hr Mon 2-3 pm.