Exam 1 Solutions

(1.) φ C^2-smooth. So φ_x and φ_y are C^1-smooth.

Let $u = \varphi_x$ and $v = -\varphi_y$. Then

$$u_x = \underbrace{\varphi_{xx} = -\varphi_{yy}}_{\text{because } \Delta\varphi = 0} = v_y \quad \checkmark$$

$$u_y = \underbrace{\frac{\partial^2\varphi}{\partial y \partial x} = \frac{\partial^2 \varphi}{\partial x \partial y}}_{\text{because } \varphi \ C^2\text{-smooth}} = -v_x \quad \checkmark$$

So u and v are C^1-smooth functions that satisfy the Cauchy-Riemann Equations, and $u+iv$ is analytic.

(2.) $\Delta u = 1 - 1 + 0 + 0 + 0 = 0$ $\checkmark$

Want v with $\quad v_x = -u_y = 2y - x - 1 \quad$ (A)

$\qquad\qquad\qquad\quad v_y = u_x = 2x + y + 1 \quad$ (B)

Use (A): $\quad v = \int 2y - x - 1 \, dx = 2xy - \frac{1}{2}x^2 - x + C(y)$

Use (B): $\quad \frac{\partial}{\partial y}\left[2xy - \frac{1}{2}x^2 - x + C(y) \right] = 2x + y + 1$

want

$$2x + C'(y) = 2x + y + 1$$

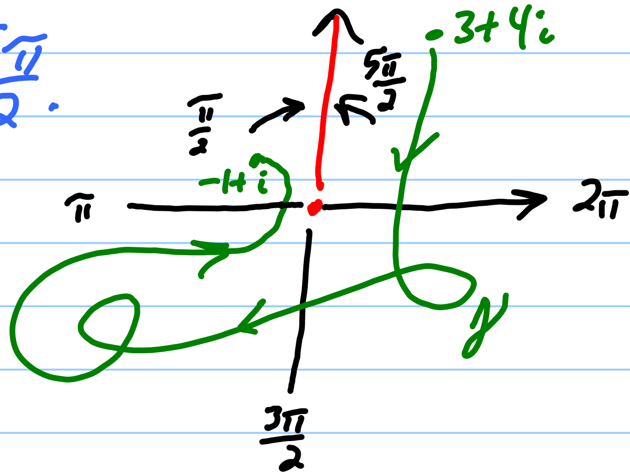
So we need $C'(y) = y + 1$. $\quad C(y) = \frac{1}{2}y^2 + y + K$

Get $v(x,y) = \underline{2xy - \frac{1}{2}x^2 - x + \frac{1}{2}y^2 + y + K}$

where K is a constant. Take $K=0$ for shortest answer.

(3.) Define $\log z = \text{Ln}|z| + i\theta$ where

$\theta \in \arg z$, $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$.



$$\int_\gamma \frac{1}{z}\, dz = \int_\gamma \frac{d}{dz} \log z \, dz$$

$$= \log(-1+i) - \log(3+4i)$$

$$= \Big(\text{Ln} \underbrace{|-1+i|}_{\sqrt{2}} + i\frac{3\pi}{4}\Big) - \Big(\text{Ln}\underbrace{|3+4i|}_{5} + i\big(\underbrace{\text{Tan}^{-1}\tfrac{4}{3}} + 2\pi\big)\Big)$$

Principal arc tan between 0 and $\frac{\pi}{2}$.

$$= \underline{\text{Ln}\frac{\sqrt{2}}{5} - i\left(\frac{5\pi}{4} + \text{Tan}^{-1}\frac{4}{3}\right)}$$

(4.) Use higher order Cauchy Integral Formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$f(z) = e^{i5z}$

$n+1 = 4$ $\boxed{n=3}$

$$\int_C \frac{e^{i5z}}{(z-a)^4}\, dz = \frac{2\pi i}{3!} f^{(3)}(a) = \frac{2\pi i}{6}(i5)^3 e^{i5a}$$

$$= \frac{125\pi}{3} e^{i5a}$$

for a inside C. But the Cauchy Theorem implies the integral is zero for $|a|>1$. So

$$\underline{I}\left(\frac{i}{3}\right) = \underline{\frac{125\pi}{3} e^{-5/3}} \quad \text{and} \quad \underline{I(3i) = 0.}$$

$I(a)$ is analytic on the unit disc because it is a constant times the third derivative of an analytic function. All derivatives of an analytic function are analytic.

5. $z = x + iy$ $\quad z^2 = (x^2 - y^2) + i\,2xy$

$$|z^2| = |z|^2 = x^2 + y^2$$

$$e^{z^2} = e^{x^2-y^2} \underbrace{e^{i\,2xy}}_{\text{modulus}=1}. \qquad \text{So } |e^{z^2}| = e^{x^2-y^2}.$$

e^t is strictly increasing. So

$$\underbrace{e^{x^2-y^2}}_{|e^{z^2}|} \le e^{x^2} \le \underbrace{e^{x^2+y^2}}_{e^{|z^2|}}$$

Equality happens exactly when $y=0$, i.e., z is real valued.

e^{z^2} does not $\to \infty$ as $z \to \infty$ because

$$e^{(iy)^2} = e^{-y^2} \longrightarrow 0 \quad \text{as } y \to \pm\infty.$$