Exam 1 Solutions

1. a) $(re^{i\theta})^3 = \sqrt{2}\, e^{i\pi/4}$. So $r^3 = 2^{1/2}$, $3\theta = \frac{\pi}{4} + 2n\pi$

$r = 2^{1/6}$, $\theta = \frac{\pi}{12} + \frac{2n\pi}{3}$, $n \in \mathbb{Z}$

$$z = \underline{2^{1/6} e^{i\pi/12}},\ \underline{2^{1/6} e^{i(\pi/12 + 2\pi/3)}},\ \underline{2^{1/6} e^{i(\pi/12 + 4\pi/3)}}$$

b) $e^{x+iy} = e^x e^{iy} = i = 1\cdot e^{i\pi/2}$. So $e^x = 1$, $y = \frac{\pi}{2} + 2n\pi$

$x = \ln 1 = 0$

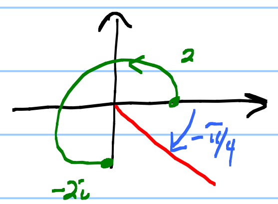
$$z = \underline{i\left(\tfrac{\pi}{2} + 2n\pi\right)},\ n \in \mathbb{Z}$$

2. $$\left.\begin{array}{c}\frac{\partial v_1}{\partial x}\\ \frac{\partial v_2}{\partial x}\end{array}\right\} = -\frac{\partial u}{\partial y} \qquad \left.\begin{array}{c}\frac{\partial v_1}{\partial y}\\ \frac{\partial v_2}{\partial y}\end{array}\right\} = \frac{\partial u}{\partial x} \qquad \text{So } \nabla v_1 \equiv \nabla v_2.$$

$\nabla(v_1 - v_2) \equiv 0$ on $\underline{\text{domain}}$ $\Omega \implies v_1 - v_2 \equiv c$, a const. ✓

3. $\gamma: z(t) = 2e^{it}$, $0 \le t \le \frac{3\pi}{2}$. $z'(t) = 2ie^{it}$

a) $$\int_\gamma \bar{z}\, dz = \int_0^{3\pi/2} \overline{(2e^{it})}\ \underbrace{(2ie^{it})}_{z'(t)}\, dt = 4i \int_0^{3\pi/2} \underbrace{e^{-it} e^{it}}_{1}\, dt = 4i\left(\frac{3\pi}{2}\right)$$
$$= \underline{6\pi i}$$

b) 

$$\int_\gamma \frac{1}{z}\, dz = \int_\gamma \frac{d}{dz}\left(\log_{-\frac{\pi}{4}} z\right) dz$$

$$= \log_{-\pi/4}(-2i) - \log_{-\pi/4}(2) = \left[\ln 2 + i\frac{3\pi}{2}\right] - \left[\ln 2 + i0\right] = \underline{i\,\frac{3\pi}{2}}$$

4. $e^{f(z)}$ is entire and $\left|e^{\overbrace{u+iv}^{f}}\right| = e^u = e^{\operatorname{Re} f} < e^0$ on $\mathbb{C}$,

so e^f is a $\underline{\text{bounded}}$ entire function. Liouville's $\implies e^f \equiv c$, const.

$e^u = |e^f| = |c|$. So $u = \ln|c| \equiv$ const. C-R Eqns $\implies \nabla v \equiv 0 \implies$

v constant. So $f = u + iv$ is constant.

5. $$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

Mult by $(z-i)(z+i)$: $0\cdot z + 1 = A(z-i) + B(z+i) = \overbrace{(A+B)}^{=0} z + i\overbrace{(-A+B)}^{=1}$

$A+B=0 \leftarrow A=-B$

$i(-A+B)=1$

$2B = 1/i$

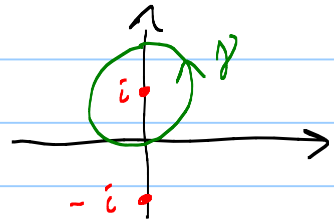
$\boxed{B=\frac{-i}{2}}$ $\boxed{A=\frac{i}{2}}$

a) $\displaystyle\int_\gamma \frac{e^{iz}}{z^2+1}\,dz = \frac{i}{2}\int_\gamma \frac{e^{iz}}{z-(-i)}\,dz + \frac{(-i)}{2}\int_\gamma \frac{e^{iz}}{z-i}\,dz$

Cauchy Int Formula

$= \pi i^2 e^{i(-i)} - \pi i^2 e^{i\cdot i} = \pi\left(-e+e^{-1}\right)$

b)



$\displaystyle\int_\gamma \underbrace{\left(\frac{e^{iz}}{z+i}\right)}_{f(z)} \cdot \frac{1}{z-i}\,dz = 2\pi i\, f(i)$

$\displaystyle = 2\pi i\,\frac{e^{i(i)}}{(i)+i} = \frac{\pi}{e}$

c) $\displaystyle\int_\gamma \frac{e^{iz}}{z^2+1}\,dz = 0$ because integrand is analytic inside and on γ, by Cauchy's Theorem.

6. $\displaystyle\left|f^{(N+1)}(a)\right| \le \frac{(N+1)!\,\max_{|z-a|=r}|f(z)|}{r^{N+1}} \le \frac{(N+1)!\,(r+|a|)^N}{r^{N+1}} \leftarrow |f(z)| \le c|z|^N$

$z=a+re^{it}$ on $C_r(a)$: $|z| \le |a|+r$

$\longrightarrow 0$ as $r\to\infty$. So $f^{(N+1)}(a)=0$ for all $a\in\mathbb{C}$.

$f^{(N+1)} \equiv 0 \implies f$ is a polynomial of degree N or less.

($f^{(N+1)} \equiv 0 \implies f^{(N)} = \text{const} \implies f^{(N-1)} = Az+B$

$\implies f^{(N-2)} = az^2+bz+c \implies \cdots$ f = poly of degree N (or less if coeff in front are zero).