MATH 425/525, Exam 1

(20) **1.** a) Using the notation $f(x+iy) = u(x,y) + iv(x,y)$, write down the Cauchy-Riemann equations and state exactly what is needed to deduce that f is an analytic function on a domain Ω.

b) State Liouville's theorem.

(20) **2.** Suppose $f(z)$ is an entire function. Show that if Re $f(z) < 0$ for all z, then f must be a constant function.

Hint: Write $f = u + iv$. Consider the modulus of $e^{f(z)}$.

(20) **3.** Suppose that γ is any curve that starts at $1-i$ and ends at $2+2i$ and avoids the non-negative real axis $\{t : 0 \leq t\}$. Compute

$$I_1 = \int_\gamma \frac{1}{z}\, dz \qquad \text{and} \qquad I_2 = \int_\gamma \frac{1}{z^2}\, dz.$$

Explain your reasoning.

(20) **4.** Compute

$$I = \int_\gamma \frac{e^{iz}}{z^2+1}\, dz$$

where

a) γ is the counterclockwise circle of radius two about the origin.

Hint: Partial fractions: $\dfrac{1}{z^2+1} = \dfrac{1}{(z-i)(z+i)} = \dfrac{i/2}{z+i} - \dfrac{i/2}{z-i}$.

b) γ is the counterclockwise circle of radius one about i.

c) γ is the counterclockwise circle of radius one-half about the origin.

(20) **5.** Compute

$$I = \int_\gamma \frac{e^{2z}}{(z-3)(z-1)^2}\, dz$$

where γ is the counterclockwise circle of radius two about the origin.

Exam 1 solutions

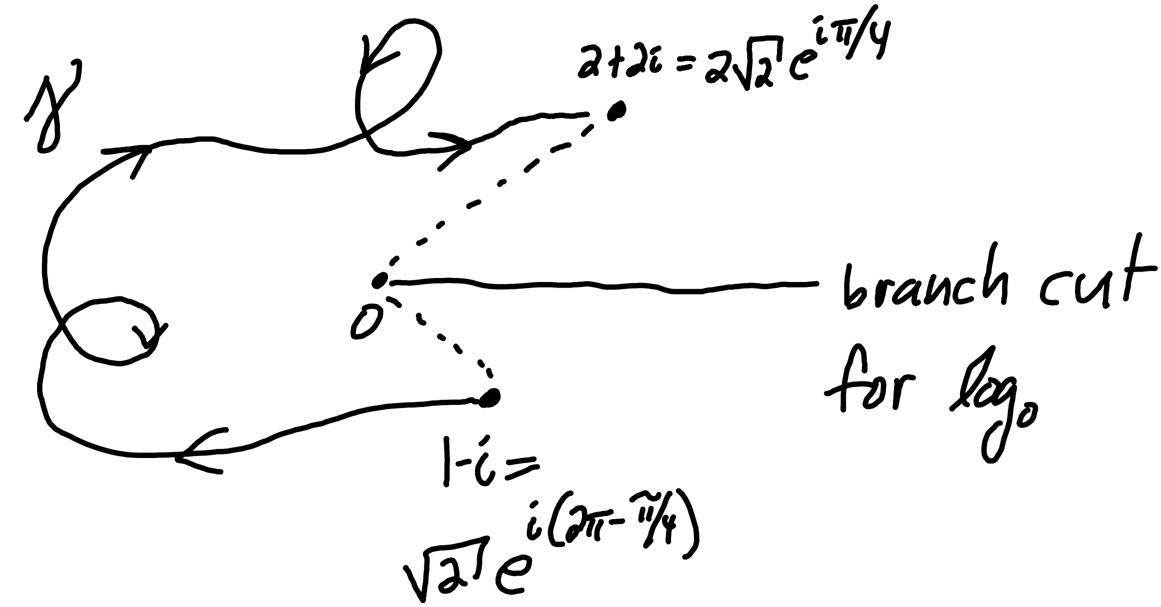
1. a) If u and v are $\mathcal{C}^1$-smooth on Ω and
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases},$$ then f is analytic.

b) Liouville's theorem: A bounded entire function must be constant.

2. Write $f = u + iv$. $|e^f| = |e^{u+iv}| = |e^u e^{iv}| = \underbrace{|e^u|}_{e^u} \cdot \underbrace{|e^{iv}|}_{1} = e^u < 1$ if $u = \operatorname{Re} f < 0$. So

e^f is a bounded entire fcn. Liouville's $\Rightarrow e^f \equiv c$, a constant. So $e^u = |e^f| \equiv |c|$ and $u \equiv \operatorname{Ln}|c|$ is constant. C-R Eqns show that
$$\begin{cases} v_x = -u_y \equiv 0 \\ v_y = u_x \equiv 0 \end{cases}.$$ So $\nabla v \equiv 0$ on $\mathbb{C} \Rightarrow v \equiv$ const too. So $f = u + iv \equiv$ const.

3.



$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left(\log_0 z\right) dz = \log_0(2+2i) - \log_0(1-i)$$

$$= \left[\operatorname{Ln} 2\sqrt{2} + i\frac{\pi}{4}\right] - \left[\operatorname{Ln}\sqrt{2} + i\left(2\pi - \frac{\pi}{4}\right)\right]$$

$$= \operatorname{Ln} 2 - i\frac{3\pi}{2}$$

$$\int_\gamma \frac{1}{z^2}\,dz = \int_\gamma \frac{d}{dz}\left(-\frac{1}{z}\right) dz = \left[-\frac{1}{z}\right]_{1-i}^{2+2i}$$

$$= -\frac{1}{2+2i} + \frac{1}{1-i} \longleftarrow \text{ans.}$$

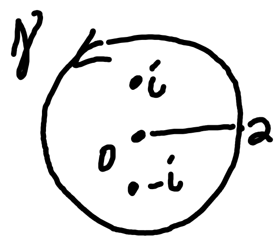
$$= -\frac{(2-2i)}{2^2+2^2} + \frac{1+i}{1^2+1^2} = -\frac{1}{4} + \frac{i}{4} + \frac{1}{2} + \frac{i}{2} = \frac{1}{4} + i\frac{3}{4}$$

4. $$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

mult by $(z-i)(z+i)$: $1 = A(z-i) + B(z+i)$

$$= \underbrace{(A+B)}_{0} z + i\underbrace{(-A+B)}_{1}$$

Get $A = \frac{i}{2}$, $B = \frac{-i}{2}$.

a) 

$$\int_\gamma \frac{e^{iz}}{z^2+1}\,dz = \int_\gamma \frac{\frac{i}{2}e^{iz}}{z-(-i)} - \frac{\frac{i}{2}e^{iz}}{z-i}\,dz$$

$$= 2\pi i\left(\frac{i}{2}e^{iz}\Big|_{z=-i} - \frac{i}{2}e^{iz}\Big|_{z=i}\right)$$

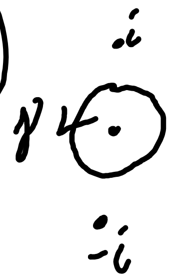
$$= 2\pi i\left(\frac{i}{2}\right)\left(e^{-i^2} - e^{i^2}\right) = -2\pi\,\frac{(e-e^{-1})}{2} \leftarrow \text{ans.}$$

$$= -2\pi \sinh 1$$

b) 

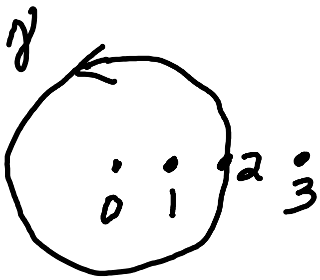
$$\int_\gamma \frac{e^{iz}/(z+i)}{z-i}\,dz = 2\pi i\left(\frac{e^{iz}}{z+i}\Big|_{z=i}\right)$$

$$= 2\pi i\,\frac{e^{i^2}}{2i} = \pi e^{-1} = \pi/e$$

4. c) 

$\int_\gamma \frac{e^{iz}}{(z+i)(z-i)}\,dz = 0$ by Cauchy's.

5. $f'(a) = \frac{1!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{1+1}}\,dz$. So



; $I = \int_\gamma \frac{e^{2z}/(z-3)}{(z-1)^2}\,dz =$

$= 2\pi i\, f'(1)$ where $f(z) = \frac{e^{2z}}{z-3}$.

$$f'(z) = \frac{(2e^{2z})(z-3) - (e^{2z})\cdot 1}{(z-3)^2}$$

$$f'(1) = \frac{2\cdot e^2(-2) - e^2}{(-2)^2} = \frac{-5}{4}e^2$$

$$I = 2\pi i\left(-\frac{5}{4}e^2\right) = \frac{-5\pi e^2 i}{2}$$