# Exam 2 solutions

(1.) $$I = \int_C \frac{\overbrace{e^{2z}}^{f(z)}}{\underbrace{2z^2+5z+2}_{g(z)}}\,dz = 2\pi i \sum_{\substack{\text{zeroes}\\ \text{of } g(z)\\ \text{inside } C}} \operatorname{Res} \frac{f(z)}{g(z)}$$

Zeroes of $g(z)$: $\dfrac{-5 \pm \sqrt{5^2-16}}{2\cdot 2} = -\dfrac{5}{4} \pm \dfrac{3}{4} = -2, \underbrace{-\tfrac{1}{2}}_{\text{inside } C}$

$$g'(z) = 4z+5$$

$g'(-\tfrac{1}{2}) = 3 \leftarrow$ not zero. $-\tfrac{1}{2}$ is a simple zero of $g$

$$\text{So } I = 2\pi i \operatorname{Res}_{-\frac{1}{2}} \frac{f}{g} = 2\pi i \frac{f(-\frac{1}{2})}{g'(-\frac{1}{2})} = 2\pi i \frac{e^{-1}}{3} = \boxed{\frac{2\pi i}{3e}}$$

(2.) $f$ has a simple pole at $a$, so the Laurent expansion of $f$ about $a$ is $\frac{A}{z-a}$ + power series in $(z-a)$. Hence

$F(z) = f(z) - \frac{A}{z-a}$ has a removable singularity at $a$. Hence, we may give $F(z)$ a value at $z=a$ to make it an entire function.

Since $f$ has a removable singularity at $\infty$, we know that $\lim_{z\to\infty} f(z)$ exists. Say $\lim_{z\to\infty} f(z) = B \in \mathbb{C}$.

Next, $\lim_{z\to\infty} F(z) = \lim_{z\to\infty} f(x) - \lim_{z\to\infty} \frac{A}{z-a} = B - 0 = B$.

Hence, $F(z)$ is a bounded entire function. So Liouville's Theorem implies that $F(z) \equiv$ const. Since $F(z) \to B$ as $z \to \infty$, that constant must be $B$. We have shown that

$$f(z) - \frac{A}{z-a} \equiv B. \quad \checkmark$$

(3.) $$\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{5n+7} z^{3n}}_{u_n}$$

Ratio test: $\left|\dfrac{u_{n+1}}{u_n}\right| =$

$$\frac{\dfrac{2^{n+1}}{5(n+1)+7}\left|z^{3(n+1)}\right|}{\dfrac{2^n}{5n+7}\left|z^{3n}\right|} = 2\cdot\frac{5n+7}{5n+12}\,|z|^3 \xrightarrow[n\to\infty]{\text{as}} 2|z|^3 \begin{array}{ll} <1 & \text{conv} \\ >1 & \text{div} \end{array}$$

So the series converges for $z$ with $|z| < \frac{1}{\sqrt[3]{2}}$ and diverges for $z$ with $|z| > 1/\sqrt[3]{2}$. So $R = \frac{1}{\sqrt[3]{2}}$.

4. $$\frac{\operatorname{Log} z}{(z^2+4)^2} = \frac{\operatorname{Log} z}{(z-2i)^2(z+2i)^2} = \frac{1}{(z-2i)^2}\Bigg[\underbrace{\frac{\operatorname{Log} z}{(z+2i)^2}}_{H(z) = A_0 + A_1(z-2i) + A_2(z-2i)^2 + \cdots}\Bigg] \leftarrow \text{analytic around } 2i$$

$$= \frac{A_0}{(z-2i)^2} + \frac{A_1}{z-2i} + A_2 + A_3(z-2i) + \cdots \text{power series} \cdots$$

$A_1 \leftarrow$ Residue

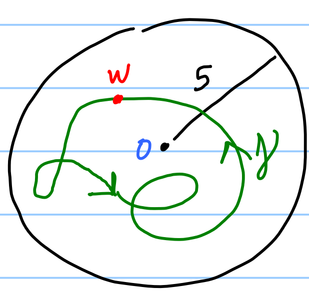
The residue at $2i$ is equal to $A_1 = \frac{H'(2i)}{1!}$ by Taylor's formula.

$$\operatorname{Res}_{2i} = \left.\frac{(z+2i)^2\cdot\frac{1}{z} - (\operatorname{Log} z)\cdot 2(z+2i)}{\left[(z+2i)^2\right]^2}\right|_{z=2i}$$

$$= \frac{(4i)^2\cdot\frac{1}{2i} - (\operatorname{Log} 2i)\cdot 2\cdot 4i}{(4i)^4} = \frac{(4i)\cdot\frac{1}{2i} - \left(\ln 2 + i\frac{\pi}{2}\right)\cdot 2}{(4i)^3}$$

$$= \frac{2 - 2\ln 2 - i\pi}{-64i} = \frac{\pi}{64} + \left(\frac{1-\ln 2}{32}\right)i$$

5.




$\bullet z$

$$|F(z)| \le \left(\max_{w\in\gamma}\left|\frac{\varphi(w)}{z-w}\right|\right)\underbrace{\operatorname{Length}(\gamma)}_{L}$$

$$\le M\cdot L\cdot \max_{w\in\gamma}\frac{1}{|z-w|}$$

Let $M = \max_{\gamma}|\varphi|$.

Say $|z| = R$. Then $|z-w| \geq \big|\underbrace{|z|}_{R} - |w|\big| \underset{\substack{\text{if} \\ R > 5 > |w|}}{=} R - |w| \underset{\substack{\text{since} \\ |w| < 5}}{>} R - 5$

So $\displaystyle \max_{w \in \gamma} \frac{1}{|z-w|} \leq \frac{1}{R-5}$ when $|z| = R > 5$.

Hence, $\displaystyle |F(z)| \leq \frac{ML}{R-5} \longrightarrow 0$ as $R = |z| \to \infty$.