MATH 425 / 525, Exam 2

(20) **1.** Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} z^{n^2}.$$

Hint: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$

(20) **2.** Let C denote the unit circle parameterized in the counterclockwise direction. Compute

$$\int_C \frac{e^{3z}}{(z - \frac{1}{2})^4} \, dz.$$

(20) **3.** Convert the integral

$$\int_0^{2\pi} \frac{dt}{2 + \sin t}$$

into a contour integral of the form $\int_C f(z)\, dz$ where f is a rational function and C is the unit circle parameterized in the counterclockwise sense. Find f, but DO NOT COMPUTE the integral.

(40) **4.** In this problem, you will compute the real integral

$$I = \int_0^{\infty} \frac{x \ln x}{x^4 + 1} \, dx$$

by integrating the complex valued function $f(z) = \dfrac{z \operatorname{Log} z}{z^4 + 1}$ (where Log denotes the principal branch of the complex log function) around the closed contour γ that follows the real axis from the origin to $R > 0$, then follows the circular arc Re^{it} as t ranges from zero to $\pi/2$, then returns to the origin via the line segment joining $Re^{i\pi/2} = iR$ to the origin, then letting $R \to \infty$.

a) Express the integral of $f(z)$ along the part of γ that follows the imaginary axis from iR to zero in terms of real integrals.

b) Use the Basic Estimate to show that the integral of $f(z)$ along the circular part of the boundary of γ tends to zero as $R \to \infty$.

c) Compute the residue of f at the point $e^{i\pi/4}$.

d) Use the Residue Theorem and let $R \to \infty$ to compute the value of I.

Exam 2 Solutions

1. Let $u_n = \frac{n^n}{n!} z^{n^2}$.

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{\left(\frac{(n+1)^{n+1}}{(n+1)!} \right)}{\left(\frac{n^n}{n!} \right)} \left| z^{(n+1)^2 - n^2} \right|$$

$$= \frac{(n+1)^n}{n^n} \left| z^{2n+1} \right| = \left(1 + \frac{1}{n}\right)^n |z|^{2n+1}$$

$\left(1+\frac{1}{n}\right)^n \to e$ as $n \to \infty$, so there is a N such that $2 < \left(1+\frac{1}{n}\right)^n < 3$ if $n > N$.

Now $\left| \frac{u_{n+1}}{u_n} \right| \le 3 |z|^{2n+1}$ if $n > N$ and this tends to zero as $n \to \infty$ if $|z| < 1$.

Ratio test $\Rightarrow$ series converges if $|z| < 1$.

$\left| \frac{u_{n+1}}{u_n} \right| \ge 2 |z|^{2n+1}$ if $n > N$ and this tends to ∞ as $n \to \infty$ if $|z| > 1$.

Ratio test $\Rightarrow$ series diverges if $|z|>1$.

So $R=1$.

2. $$\frac{e^{3z}}{(z-\frac{1}{2})^4} = \frac{a_0}{(z-\frac{1}{2})^4} + \frac{a_1}{(z-\frac{1}{2})^3} + \frac{a_2}{(z-\frac{1}{2})^2} + \frac{a_3}{z-\frac{1}{2}} + \cdots$$

Res

where $$a_3 = \frac{\frac{d^3}{dz^3}\left(e^{3z}\right)}{3!}\Bigg|_{z=\frac{1}{2}} = \frac{3^3 e^{3/2}}{3\cdot 2\cdot 1} = \frac{9}{2}e^{3/2}$$

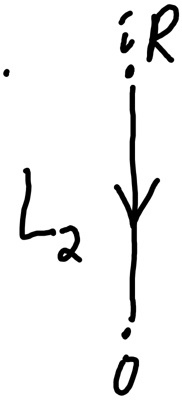
$$\int_C \frac{e^{3z}}{(z-\frac{1}{2})^4}\,dz = 2\pi i \operatorname{Res}_{\frac{1}{2}} \frac{e^{3z}}{(z-\frac{1}{2})^4}$$

$$= 2\pi i\left(\frac{9}{2}e^{3/2}\right) = \underline{\underline{9\pi i e^{3/2}}}$$

3. $$\int_0^{2\pi} \frac{1}{2+\left(\frac{e^{it}-e^{-it}}{2i}\right)} \frac{ie^{it}}{ie^{it}}\,dt$$

$$= \int_C \frac{1}{2+\frac{1}{2i}(z-\frac{1}{z})}\cdot\frac{1}{iz}\,dz$$

4. a.



$$-L_2 : z(t) = it, \quad 0 \leq t \leq R$$

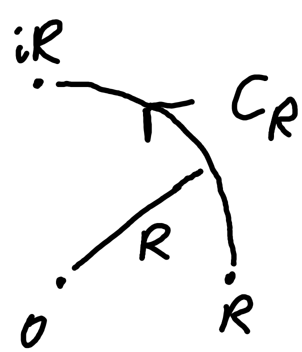
$$\int_{L_2} f(z)\,dz = -\int_{-L_2} f(z)\,dz$$

$$= -\int_0^R \frac{(it)\,\mathrm{Log}(it)}{(it)^4+1} \cdot i\,dt$$

$$= \int_0^R \frac{t\left(\mathrm{Ln}\,t + i\frac{\pi}{2}\right)}{t^4+1}\,dt$$

$$= \int_0^R \frac{t\,\mathrm{Ln}\,t}{t^4+1}\,dt + i\frac{\pi}{2}\int_0^R \frac{t}{t^4+1}\,dt$$

b.



Notice that

$$|\mathrm{Log}\,z| = |\mathrm{Ln}|z| + i\,\mathrm{Arg}\,z|$$

$$\leq \mathrm{Ln}\,R + \mathrm{Arg}\,z$$

$$\leq \mathrm{Ln}\,R + \frac{\pi}{2} \quad \text{if } |z| = R > 1.$$

Also, $|z^4+1| \geq \big||z|^4-|-1|\big|$

$$\geq R^4-1 \text{ if } |z|=R>1.$$

So the Basic estimate yields

$$\left|\int_{C_R} f(z)\,dz\right| \leq \left(\max_{C_R}\left|\frac{z\,\mathrm{Log}\,z}{z^4+1}\right|\right)\underbrace{\mathrm{Length}(C_R)}_{\frac{\pi}{2}R}$$

$$\leq \frac{R\left(\ln R+\frac{\pi}{2}\right)}{R^4-1}\cdot\frac{\pi}{2}R \quad \text{if } R>1$$

$$\longrightarrow 0 \text{ as } R\to\infty.$$

c. Let $F(z)=z\,\mathrm{Log}\,z$ and $G(z)=z^4+1$.

G has a <u>simple</u> zero at $e^{i\pi/4}$ because

$$G(e^{i\pi/4})=0 \quad \text{and} \quad G'(e^{i\pi/4})=4\left(e^{i\pi/4}\right)^3\neq 0.$$

So $\operatorname{Res}_{e^{i\pi/4}} \dfrac{F(z)}{G(z)} = \dfrac{F(e^{i\pi/4})}{G'(e^{i\pi/4})}$

$$= \frac{e^{i\pi/4}\,\operatorname{Log} e^{i\pi/4}}{4\,e^{i3\pi/4}} = \frac{\overbrace{\operatorname{Ln}|e^{i\pi/4}|}^{=0} + i\frac{\pi}{4}}{4\,\underbrace{e^{i\pi/2}}_{=i}}$$

$$= \underline{\underline{\frac{\pi}{16}}}$$

d) 

$$\left(\int_{L_1} + \int_{L_2} + \int_{C_R}\right) = 2\pi i \operatorname{Res}_{e^{i\pi/4}} f$$

$$\longrightarrow \left(\int_0^\infty \frac{t\operatorname{Ln} t}{t^4+1}\,dt\right) + \left(\int_0^\infty \frac{t\operatorname{Ln} t}{t^4+1}\,dt + i\frac{\pi}{2}\int_0^\infty \frac{t}{t^4+1}\,dt\right) = \frac{\pi^2}{8}i$$

We obtain: $\displaystyle\int_0^\infty \frac{t\ln t}{t^4+1}\,dt = 0$

$$\left(\text{and } \int_0^\infty \frac{t}{t^4+1}\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{8} = \frac{\pi}{4}\right)$$