

MATH 425 Final Exam

Each problem is worth 25 points

1. Suppose f and g are two entire functions such that g is not identically equal to zero and

$$|f(z)| \leq |g(z)|$$

for all z . Prove that $f = cg$ for some constant c with $|c| \leq 1$. (Be mindful of points where g might vanish.)

2. Compute

$$\int_0^\infty \frac{x^2}{x^4 + 1} dx$$

by integrating around a quarter circle of radius R and letting $R \rightarrow \infty$. If you state that a limit is zero, prove that it is zero. Simplify your answer.

3. Let C_1 denote the unit circle parametrized in the counterclockwise sense. Evaluate the following integrals. Explain.

a) $\int_{C_1} \frac{e^{2z}}{z - 4} dz$

b) $\int_{C_1} \frac{e^{2z}}{4z^2 + 1} dz$

c) $\int_{C_1} \frac{e^{2z}}{2z^2 - 5z + 2} dz$

d) $\int_{C_1} \frac{e^{2z}}{(z - 1/2)^5} dz$

e) $\int_{C_1} \frac{1}{z^5} dz$

4. What is the radius of convergence of the power series for $\frac{\sin z}{z^2 + 9}$ centered at $z = 4$? Explain.
5. Find a sequence of analytic functions that, when composed, maps the quarter disc $\{re^{i\theta} : 0 < r < 1, 0 < \theta < \pi/2\}$ one-to-one onto the strip $\{z : 0 < \operatorname{Re} z < 1\}$.
6. How many zeroes does the polynomial $z^4 + z^3 - z^2 + 2$ have inside the disc of radius two about the origin? Explain.
7. What is the image of the unit disc under the following mappings?
a) $T_1(z) = z/(1 - z)$ b) $T_2(z) = z/(z - i/2)$
8. Find a harmonic function on the unit disc that has boundary values on the unit circle equal to one in the first quadrant, two in the second quadrant, three in the third quadrant, and four in the fourth quadrant.