MATH 425, Practice Problems

1. Prove that if a twice continuously differentiable harmonic function vanishes on an open subset of a domain, then it must be zero on the whole domain. Hint: If u is harmonic, then $u_x - iu_y$ is analytic.

2. Suppose that f and g are analytic on a disc about a and that each has a zero of multiplicity N at a. Prove that $\lim_{z\to a} \dfrac{f(z)}{g(z)} = \dfrac{f^{(N)}(a)}{g^{(N)}(a)}$.

3. Let $f(z) = \frac{1}{1+z^2}$. Find the first three terms in the power series $\sum_{n=0}^{\infty} a_n(z-2)^n$ for f centered at $z = 2$. What is the radius of convergence of the series? Explain.

4. What is the radius of convergence of the power series for $\frac{\sin z}{z}$ centered at $z = \pi$?

5. From Saff and Snyder, know how to do p. 267: 3, p. 317: 1, and p. 325: 11.

6. Suppose $a_1, a_2, \ldots, a_n$ are points on the unit circle. For a point z on the unit circle, let $D(z)$ denote the product of the distances from z to each of the n points on the circle. Prove that there is a point z on the circle such that the product $D(z)$ is exactly equal to one. Hint: Let $p(z)$ denote the product of $(z - a_n)$ as n ranges from one to n. What is $|p(0)|$? Use the Maximum Modulus Principle.

7. State Liouville's Theorem. Show that if an entire function $f(z)$ is such that Re $f(z) < 0$ for all z, then f must be constant.

8. Suppose that f and g are analytic on a disc about a point a and suppose that g has a zero of multiplicty two at a. Show that the residue of f/g at a is equal to

$$\frac{6g''(a)f'(a) - 2g'''(a)f(a)}{3g''(a)^2}.$$

9. Assume $b_0 = 1$. Constants $b_2, b_4, b_6, \ldots$ are generated via the recursion formula

$$b_{n+2} = \frac{4(n+3)(n-5)}{7(n+1)(n+2)}\, b_n.$$

What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} b_{2n} z^{2n} = b_0 + b_2 z^2 + b_4 z^4 + \cdots ?$$

10. If the complex numbers a_n tend to zero as n tends to infinity, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is at least one.