MATH 425, Practice Problems

1. Prove that if a twice continuously differentiable harmonic function vanishes on an open subset of a domain, then it must be zero on the whole domain. Hint: If u is harmonic, then $u_x - iu_y$ is analytic.

2. Suppose that f and g are analytic on a disc about a and that each has a zero of multiplicity N at a. Prove a L'Hôpital's formula: $\lim_{z \to a} \dfrac{f(z)}{g(z)} = \dfrac{f^{(N)}(a)}{g^{(N)}(a)}$.

3. Let $f(z) = \frac{1}{1+z^2}$. Find the first few terms in the power series $\sum_{n=0}^{\infty} a_n (z-2)^n$ for f centered at $z = 2$. What is the radius of convergence of the series? Explain.

4. What is the radius of convergence of the power series for $\frac{\sin z}{z}$ centered at $z = \pi$?

5. Show that the Maximum Principle implies the Fundamental Theorem of Algebra.

6. State Liouville's Theorem. Show that if an entire function $f(z)$ is such that $\text{Re}\, f(z) < 0$ for all z, then f must be constant.

7. Suppose that f and g are analytic on a disc about a point a and suppose that g has a zero of multiplicty two at a. Show that the residue of f/g at a is equal to

$$\frac{6g''(a)f'(a) - 2g'''(a)f(a)}{3g''(a)^2}.$$

8. If the complex numbers a_n tend to zero as n tends to infinity, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is at least one.

9. Prove that Rouché's theorem implies the Open mapping theorem.

10. Find an analytic function that maps the quarter disc $\{re^{i\theta} : 0 < r < 1, \;\; 0 < \theta < \pi/2\}$ one-to-one onto the strip $\{z : 0 < \text{Re}\, z < 1\}$.

11. What is the image of the unit disc under the linear fractional transformation $T(z) = z/(1-z)$?

12. Problems from the book:

p. 267: 3
p. 317: 1
p. 325: 11
p. 430: 5, 6
p. 440: 3