Math 428

Homework 6

1. Assume that f and F are real valued continuous functions on $[-\pi,\pi]$. Prove that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)F(t)\ dt = 2a_0A_0 + \sum_{n=1}^{\infty}(a_nA_n + b_nB_n),$$

where the lowercase coefficients are the real Fourier coefficients for f (given on the sheet of Important Formulas), and the uppercase are the coefficients for F.
Hint: Write out the real Parseval's identity for $f+F$ and $f-F$ and subtract.

2. Let $\hat{f}(n)$ and $\hat{g}(n)$ denote the complex Fourier coefficients of continuous functions f and g, respectively. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\ d\theta = \sum_{n=-\infty}^{\infty} \hat{f}(n)\,\overline{\hat{g}(n)}.$$

Hints: Write out the complex Parseval's identity for $f+\lambda g$ and $f-\lambda g$ and subtract. Then pick a complex number λ of unit modulus to deduce the identity. (Or maybe pick two such complex λ's.) $z\,\bar{w} + \bar{z}\,w = 2\mathrm{Re}\,(z\,\bar{w})$

3. If γ is a C^2-smooth simple closed curve, show that $\int_\gamma x\,dy$ is equal to the area enclosed by γ. Explain how you could design a computer for NASA that could figure out the area of a region on Mars by driving a rover around it while consulting a compass and the speedometer. (You may assume that Mars has a magnetic field and is flat.)

4. Do Problem 9 on page 90 of Stein.

5. Show that the parametric equations

$$x(t) = \alpha\cos t - \beta\sin t$$
$$y(t) = \beta\cos t + \alpha\sin t$$

describe a circle centered at the origin. Hint: $x(t) + iy(t) = (\alpha + i\beta) * (?)$

6. Given $f(\theta) = \theta(\pi-\theta)$ on $[0,\pi]$, use MAPLE to find the real and complex Fourier coefficients of the odd 2π periodic extension of f and show that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$