Review for Midterm exam   Exam in class Wed, March 5

1. Explain why

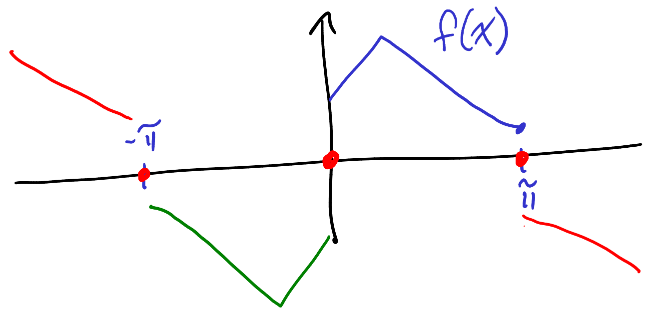
symmetric interval about 0 $\longrightarrow$
$$\int_{-\pi}^{\pi} \underbrace{\overset{\text{even}}{e^{-x^2}}\ \overset{\text{odd}}{\sin 3x}}_{\text{odd}}\, dx = 0$$

without trying to find an antiderivative (which is impossible).

f(x) is odd fcn: No cosines terms or constant term in F.S.

even : No sine terms.

Cooking up Fourier sine series:



Take odd extension

Full Fourier series of extension restricted to $[0, \pi]$ is F. sine series

2π-periodic extension ← Full Fourier series converges to this

1. Suppose that $f(x)$ is a C^1-smooth function on $[a,b]$. Use integration by parts to show that

$$I = \int_a^b f(x)\cos(Nx)\,dx$$

tends to zero as the positive integer N tends to infinity.

$$\begin{cases} u = f(x) \\ dV = \cos Nx\,dx \end{cases} \qquad du = f'(x)\,dx$$

$$V = \int \cos Nx\,dx$$

$$V = \frac{1}{N}\sin Nx$$

$$I = uv\Big|_a^b - \int_a^b v\,du$$

$$= \underbrace{f(x)}_{\text{bounded}}\ \underbrace{\frac{1}{N}}_{\frac{1}{N}\to 0}\ \underbrace{\sin Nx}_{|\,|\le 1}\Big|_a^b - \underbrace{\frac{1}{N}}_{\frac{1}{N}\to 0}\int_a^b \underbrace{\sin Nx}_{|\,|\le 1}\ \underbrace{f'(x)}_{\text{bndd}}\,dx$$

2. Estimate the integral in problem one in terms of the maximum value M of $|f(x)|$ on $[a, b]$.

$$\left| \int_a^b f(x) \cos Nx \, dx \right| \le \int_a^b \underbrace{|f(x)|}_{\le M} \, \underbrace{|\cos Nx|}_{\le 1} \, dx$$

$$\le \int_a^b M \, dx = M(b-a)$$

3. Show that the integral in problem one goes to zero as $N \to \infty$ when f is merely assumed to be continuous by approximating f by a polynomial and then using your results from problem one and two.

Weierstraß thm: Given $\varepsilon > 0$, there is a poly $P(x)$ such that $|f(x) - P(x)| < \frac{\varepsilon}{2(b-a)}$ on $[a,b]$.

$$(*) \quad \left| \int_a^b f(x) \cos Nx \, dx \right|$$

$$= \left| \int_a^b \left(f(x) - P(x)\right) \cos Nx \, dx + \int_a^b P(x) \cos Nx \, dx \right|$$

$$\leq \underbrace{\left| \int_a^b \left(f(x) - P(x)\right) \cos Nx \, dx \right|}_{\leq \underbrace{\frac{\varepsilon}{2(b-a)} \cdot (b-a)}_{\frac{\varepsilon}{2}}} + \underbrace{\left| \int_a^b P(x) \cos Nx \, dx \right|}_{\to 0 \text{ as } N \to \infty}$$

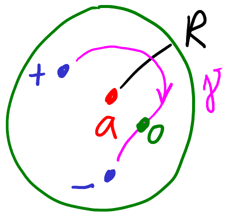
Can make $< \frac{\varepsilon}{2}$ by taking $N > N_0$.

[$P(x)$ is C^1-smooth!]

See $I \to 0$ as $N \to \infty$.

5. Prove that harmonic functions cannot have isolated zeroes.

Key: Ave property plus fact:



a isolated zero of harmonic fcn u.

$\exists$ disc $D_R(a)$ so that a is the only zero of u in $D_R(a)$

$\gamma : (x(t), y(t)) ; \alpha \le t \le \beta$

$u(x(t), y(t))$

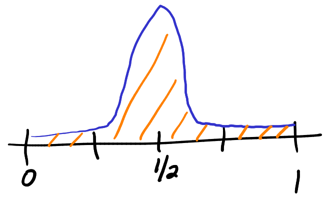
IMT $\Rightarrow$ $\exists$ t_0 where $u(x(t_0), y(t_0)) = 0$, some $\alpha < t_0 < \beta$. ↯

$$\underbrace{u(a)}_{=0} = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(a+\rho e^{i\theta})}_{\substack{\text{always } + \\ \text{or always } -}}\, d\theta}_{\int \text{ is } >0 \text{ or } <0.} \qquad 0<\rho<R.$$

↯

4. Suppose $P(x)$ is a positive continuous function on $[0,1]$ such that $P(1/2) = 10^4$, and $P(x) < 1/100$ when x is in $[0,1/4] \cup [3/4,1]$, and

$$\int_0^1 P(x)\, dx = 1.$$



If f is a continuous function such that $|f(x)| < 17$ on $[0,1]$, but $|f(x)| < 1/3$ on $[1/4, 3/4]$, estimate how large

smallish at bump of P

$$I = \left| \int_0^1 P(x) f(x)\, dx \right|$$

could be. What is your best estimate for the bound if the assumption that P is positive is dropped?

$$I = \left| \left(\int_0^{1/4} + \int_{3/4}^1 \right) \underbrace{P(x)}_{P\text{ small}} f(x)\, dx + \int_{1/4}^{3/4} P(x) \underbrace{f(x)}_{\text{smallish}} dx \right|$$

$$\left| \int_0^{1/4} + \int_{3/4}^1 \right| \le \left(\frac{1}{100}\right) \cdot 17 \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\text{lengths of intervals}} \leftarrow \frac{17}{200}$$

$$\left| \int_{1/4}^{3/4} P(x) \underbrace{f(x)}_{|\ | < \frac{1}{3}} dx \right| \le \int_{1/4}^{3/4} \underbrace{P(x)}_{|P(x)| = P(x)} \cdot \frac{1}{3}\, dx$$

$$\le \frac{1}{3} \underbrace{\int_0^1 P(x)\, dx}_{=1} \qquad \le \frac{1}{3}$$

Ouch! Without knowing $P(x)$, get $\overset{?}{<}\ 10^4 \cdot \frac{1}{3}$ or maybe even bigger!

Oops! Wish I had made problem

$$P\left(\tfrac{1}{2}\right) = 10^4$$

$$|P(x)| \le 10^4 \text{ on } [0,1].$$

(Need $|P(x)| < \frac{1}{100}$ on $[0, 1/4] \cup [3/4, 1]$ too.)

2. Given a piecewise C^1-smooth real valued function $g(x)$ on $[-\pi, \pi]$, what real value of the constant A makes

$$\int_{-\pi}^{\pi} |g(x) - A|^2 \, dx$$

Trig poly of order $N=0$.

as small as possible. Explain.

Big fact: The best L^2-approximation to g via trig polys of order N is the Fourier series trig poly

$$a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx \right)$$

Fourier coeff

a_0 = ave of g on $[-\pi, \pi]$

In fact, if trig poly of order N is different than this, then L^2-integral is strictly bigger.

4. Find a closed expression for the sum

$$1 - \cos\theta + \cos 2\theta - \cos 3\theta + \cdots + (-1)^N \cos N\theta$$

that does not have lengthy sums and contains real functions and numbers only. Hint: Replace z by $-z$ in the famous identity

$$1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}$$

and use Euler's and DeMoivre's formulas. (No need to use trig identities to try to simplify answer.)

Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$

DeMoivre's formula: $\left(e^{i\theta}\right)^N = e^{iN\theta}$

$$1 - \cos\theta + \cos 2\theta - \cdots + (-1)^N \cos N\theta$$

$$= \operatorname{Re}\left[1 + z + z^2 + \cdots + z^N\right] \quad \text{where } z = -e^{i\theta}$$

$$= \operatorname{Re}\left[\frac{1 - z^{N+1}}{1 - z}\right] = \operatorname{Re}\left[\frac{1 - (-e^{i\theta})^{N+1}}{1 - (-e^{i\theta})}\right]$$

$$= \operatorname{Re}\left[\underbrace{\frac{\left[1 - (-1)^{N+1}\right]\cos(N+1)\theta - i(-1)^{N+1}\sin(N+1)\theta}{(1 + \cos\theta) + i\sin\theta}}\right]$$

$$\frac{A + Bi}{C + Di} \cdot \frac{C - Di}{C - Di}$$

$$\underbrace{\frac{(AC + BD)}{C^2 + D^2}}_{\text{Real part}} + i\,\frac{(BC - AD)}{C^2 + D^2}$$