MA 428 Fourier Analysis

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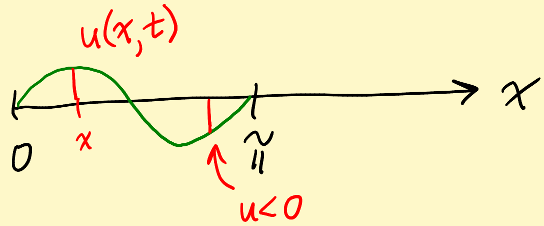
Power series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Fourier series: $f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$

Bernouli vs Bernouli : Vibrating string problem



Harpsichord problem

Initial conditions

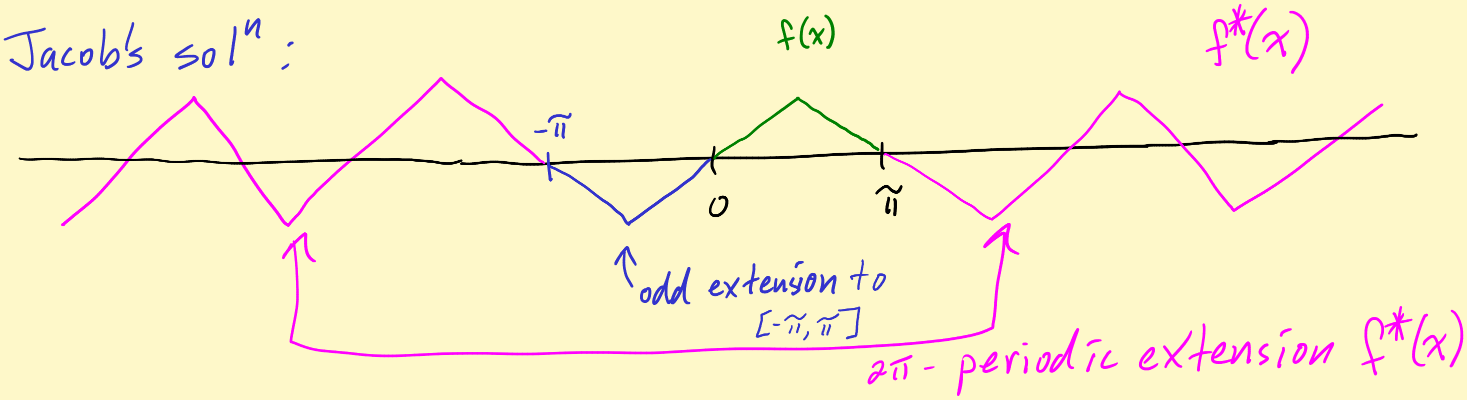
$u(x,0) = \text{known}$

$\frac{\partial u}{\partial t}(x,0) \equiv 0$ ← plucked, not hammered

Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Boundary conditions: $\begin{cases} u(0,t) \equiv 0 \\ u(\pi,t) \equiv 0 \end{cases}$

Jacob's sol^n:



$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) \equiv 0 \end{cases}$$

wave moving left moving right

Sol^n: $u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct) \right]$ ← winner!

Johann's Sol^n: Try $u(x,t) = X(x)T(t)$

(separation of variables)

Assume $c=1$.

Plug into PDE: $\frac{\partial^2}{\partial t^2}\left[X(x)T(t)\right] \overset{\text{want}}{=} \frac{\partial^2}{\partial x^2}\left[X(x)T(t)\right]$

$$X(x)T''(t) = X''(x)T(t)$$

Let $x = \frac{\pi}{2}$.

See $\frac{T''}{T}$ = const.

So $\frac{X''}{X}$ const too!

$$\boxed{\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}} = \lambda$$

λ a const.

Get 2 ODE's: $\begin{cases} X'' - \lambda X = 0 \\ T'' - \lambda T = 0 \end{cases}$

Initial conditions hopeless now. Save for later.

Bound cond: $\begin{cases} u(0,t) = X(0)T(t) \overset{\text{want}}{=} 0 \\ u(\pi,t) = X(\pi)T(t) = 0 \end{cases}$

Hmmm. Could take $T(t) \equiv 0$. Dumb. Then $u(x,t) \equiv 0$. The zero sol^n. String justs sits there.

Need to take $\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$ also boundary conditions!

Famous problem: Sturm-Liouville problem

$$\boxed{\begin{gathered} X'' - \lambda X = 0 \\ \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \end{gathered}}$$

t- problem

$T'' - \lambda T = 0$

save it for after we solve X prob.

MA 262: $X'' - \lambda X = 0$ is a 2nd order linear ODE with constant coeff

Try $X(x) = e^{rx}$.

Plug in: $[e^{rx}]'' - \lambda [e^{rx}] = 0$ want

$$r^2 e^{rx} - \lambda e^{rx} = 0$$

$$\underbrace{(r^2 - \lambda)}_{\text{need } = 0} \underset{\text{never zero}}{e^{rx}} = 0$$

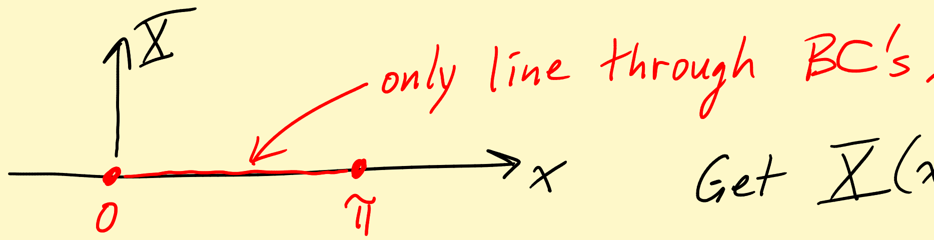
$$\boxed{r^2 = \lambda}$$

Case $\lambda = 0$:

$$X''(x) = 0$$

$$X'(x) = A$$

$$X(x) = Ax + B \leftarrow \text{line}$$



Get $X(x) \equiv 0$.

Get $u \equiv 0$ again.

Case $\lambda > 0$. Write $\lambda = k^2$ where $k > 0$.

$$r^2 = \lambda = k^2$$

$$r = \pm k.$$

Get $\underline{X_1(x) = e^{kx}}$, $\underline{X_2(x) = e^{-kx}}$ Lin. Indep.

Gen^l Sol^n $X(x) = c_1 e^{kx} + c_2 e^{-kx}$

$$X(0) = c_1 \cdot 1 + c_2 \cdot 1 \overset{\text{want}}{=} 0 \leftarrow \text{Get } \boxed{c_2 = -c_1}$$

$$X(\pi) = c_1 e^{k\pi} - c_1 e^{-k\pi} \overset{\text{want}}{=} 0$$

$$= c_1 \big(\underbrace{\underset{>1}{e^{k\pi}} - \underset{<1}{e^{-k\pi}}}_{>0}\big) = 0$$

Get $c_1 = 0$.

$c_2 = -c_1 = 0$.

Darn! $u \equiv 0$ again