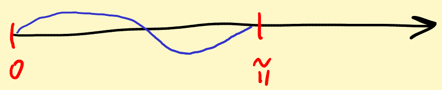
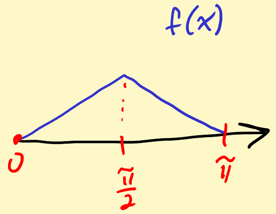
Lesson 2 Solution of the harpsichord problem



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

BC $\begin{cases} u(0,t) \equiv 0 \\ u(\pi,t) \equiv 0 \end{cases}$

IC $\begin{cases} u(x,0) = \end{cases}$



$\frac{\partial u}{\partial t}(x,0) \equiv 0$ ← plucked

Johann Bernoulli's sol^n: Try $u(x,t) = X(x)T(t)$

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

$$\boxed{\begin{aligned} &X'' - \lambda X = 0 \\ &\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \end{aligned}} \qquad T'' - \lambda T = 0$$

Case $\lambda = 0$ Only get $X(x) \equiv 0$, $u(x,t) \equiv 0$.

Case $\lambda > 0$ Same thing: $u(x,t) \equiv 0$.

Case $\lambda < 0$ Write $\lambda = -k^2$. $(k > 0)$.

Try e^{rx}: $r^2 e^{rx} - \lambda e^{rx} \overset{\text{want}}{=} 0$

$$\underbrace{(r^2 - \lambda)}_{\uparrow \text{ characteristic eqn: } r^2 - \lambda = 0} e^{rx} = 0$$

Need $r^2 = \lambda = -k^2$. Ouch. $r = \pm ki$

Euler: $\underbrace{e^{ikx}}_{= u(x)+iv(x)}$, e^{-ikx} are two complex sol^{n}s. (They really work)

$$\boxed{e^{ikx} = \cos kx + i \sin kx}$$

$$Lf = f'' - \lambda f$$

$$L[u+iv] = (u''+iv'') - \lambda(u+iv) = \underbrace{(u''-\lambda u)}_{=0} + i\underbrace{(v''-\lambda v)}_{=0} = 0$$

Fact From one complex sol^n $u+iv$, get two real sol^{n}s: u, v.

Get real sol^{n}s: $X(x) = c_1 \cos kx + c_2 \sin kx$

Claim: This is the gen^l sol^n because I can solve the ODE with any and all initial conditions.

Write out $\begin{cases} X(0) = A \\ X'(0) = B \end{cases}$

want

$$\begin{cases} c_1 \cos k\cdot 0 + c_2 \sin k\cdot 0 = A \\ -c_1 k \sin k\cdot 0 + c_2 k \cos k\cdot 0 = B \end{cases}$$

$$\underbrace{\begin{bmatrix} \cos k\cdot 0 & \sin k\cdot 0 \\ -k \sin k\cdot 0 & k\cos k\cdot 0 \end{bmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\det = k\left(\underbrace{\cos^2 k\cdot 0 + \sin^2 k\cdot 0}_{1}\right) = k \neq 0.$$

Wronskian! $\neq 0$.

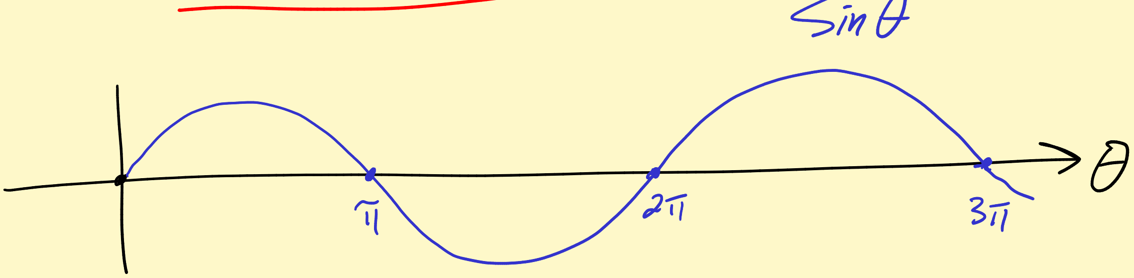
$\boxed{X(x) = c_1 \cos kx + c_2 \sin kx}$ is gen^l sol^n!

BC's: $X(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 \overset{\text{need}}{=} 0$ $\boxed{c_1 = 0}$

$X(\pi) = c_2 \sin k\pi \overset{\text{need}}{=} 0$

Hmmm. Don't want $c_2 = 0$ too. (Get zero sol^n again.)

Need $\boxed{\sin k\pi = 0}$



Aha! Need $k\pi = n\pi$ $n = 1, 2, 3, 4, \dots$

Get non-zero sol^n $X_n(x)$ when $k = n \in \mathbb{N}$. $X_n(x) = \sin nx$

Now get $T_n(t)$: $T'' + n^2 T = 0$ $(\lambda = -n^2)$

$$T(t) = A_n \cos nt + B_n \sin nt$$

Get $u_n(x,t) = X_n(x) T_n(t) = \sin nx \left(A_n \cos nt + B_n \sin nt\right)$

Time to worry about initial condition.

"Supperposition principle" holds!

PDE $Lu = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ is a <u>linear</u> PDE. (homogeneous)

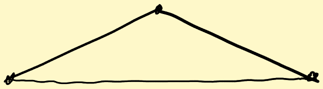
$$L[\underbrace{c_1u_1 + c_2u_2}_{\text{get new sol}^n \text{ from sol}^n\text{'s } u_1,\, u_2}] = c_1 \underbrace{L[u_1]}_{=0} + c_2 \underbrace{L[u_2]}_{=0}$$

<u>Also</u>, the boundary conditions are "homogeneous":

If $\begin{cases} u_j(0,t)=0 \\ u_j(\pi,t)=0 \end{cases}$ for $j=1,2$, then

$c_1u_1 + c_2u_2$ satisfy the BC too.

Hmmm. $\sum_{n=1}^{N} X_n(x)T_n(t)$ satisfies PDE and BC's.

Can't get pluck shape  for initial shape.

Big idea: Let $N \to \infty$. Hope it works.

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(A_n \cos nt + B_n \sin nt\right) \\ \frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin nx \left(-nA_n \sin nt + nB_n \cos nt\right) \end{cases}$$

$$\text{IC} \begin{cases} u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \overset{\text{want}}{\equiv} f(x) \\ \frac{\partial u}{\partial t}(x,0) = \sum nB_n \sin nx \equiv 0 \end{cases}$$

Take B_n's $=0$.

Find A_n so that $f(x) = \sum_{n=1}^{\infty} A_n \sin nx$

Key $\sin nx$ are "orthogonal functions" on $[0,\pi]$.

$$\langle \sin nx, \sin mx \rangle = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$n=m$: $\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2nx) \, dx = \frac{\pi}{2}$

Idea $\left[\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_N \vec{u}_N \right] \cdot \vec{u}_m$

$$\vec{v} \cdot \vec{u}_m = 0 + 0 + \cdots + c_m \vec{u}_m \cdot \vec{u}_m + 0 + \cdots + 0$$

$$c_m = \frac{\vec{v} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m} \leftarrow \text{determined!}$$

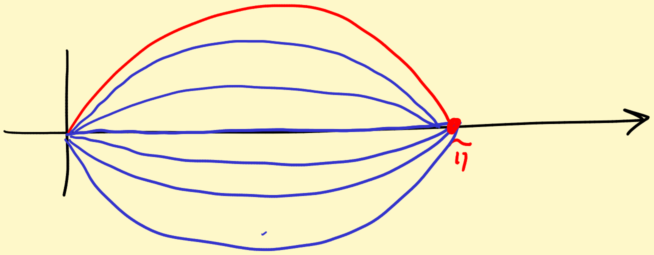
$$\int_0^{\pi} f(x) \cdot \sin mx \, dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi} \sin nx \cdot \sin mx \, dx}_{\substack{= 0 \quad n \neq m \\ = \frac{\pi}{2} \quad n = m}}$$

$$\int_0^{\pi} f(x) \sin mx \, dx = A_m \cdot \frac{\pi}{2}$$

$$\boxed{A_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx}$$

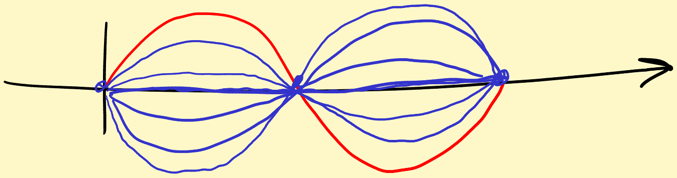
$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt \quad \text{is "solution"}$$

n=1



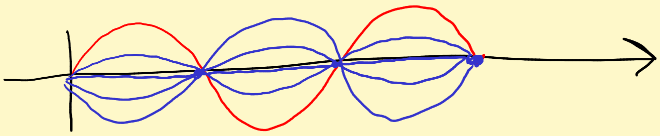
$\sin x \cos t$

n=2



$\sin 2x \cos 2t$

n=3



$\sin 3x \cos 3t$