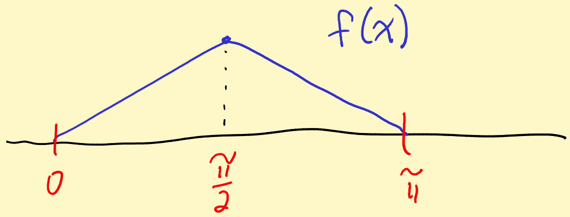
Lesson 3 Fourier series and music

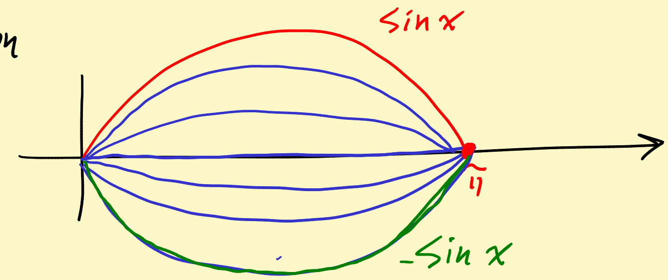


$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Sol^n: $u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt$ where $A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

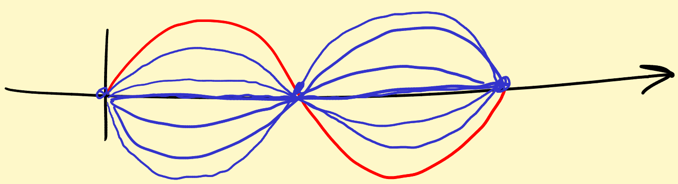
Modes of oscillation

$n=1$



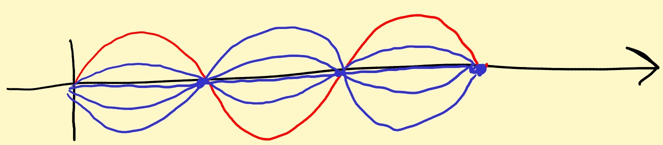
$\sin x \cos t$

$n=2$

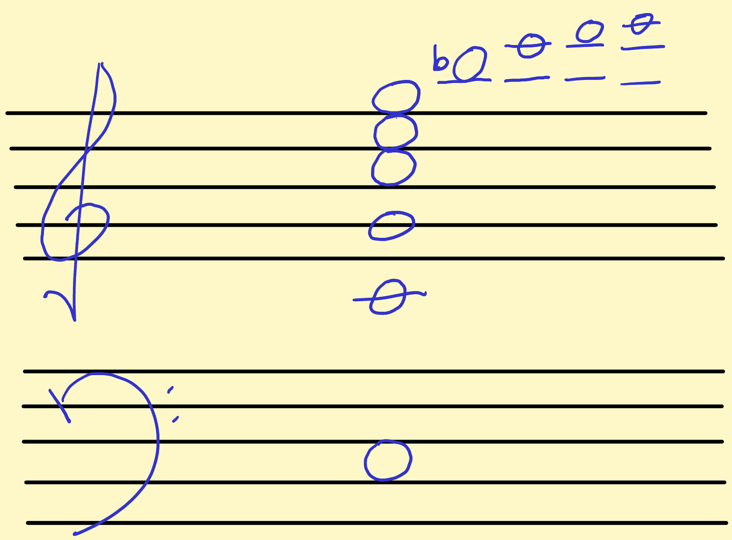


$\sin 2x \cos 2t$

$n=3$



$\sin 3x \cos 3t$



$n=7$ ← first "ouch" out of tune with C

$n=6$

$n=5$ E

$n=4$ C

$n=3$ G

$n=2$ C

$n=1$ C

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \underset{u}{x}\, \underbrace{\sin nx\, dx}_{dv} + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$$

$$A_n = \frac{4}{\pi}\left(\underset{n=1}{\frac{1}{1^2}},\ \underset{n=2}{0},\ \underset{n=3}{\frac{-1}{3^2}},\ 0,\ \frac{1}{5^2},\ 0,\ -\frac{1}{7^2},\ 0,\ \cdots\right)$$

$$u(x,t) = \frac{4}{\pi}\left(\frac{1}{1^2}\sin x\cos t - \frac{1}{3^2}\sin 3x\cos 3t + \frac{1}{5^2}\sin 5x\cos 5t - \cdots\right)$$

$$u(x,0) = \frac{4}{\pi}\left(\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right)$$

$$\stackrel{?}{=} f(x)$$

Yes! MAPLE says so!

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)x = f(x)$$

MA 341: <u>Weierstraß M-test</u>: $\varphi_n(x)$ continuous on $[a,b]$ and $|\varphi_n(x)| \le M_n$ there <u>and</u> $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty}\varphi_n(x)$ converges <u>uniformly</u> to a continuous fcn $\varphi(x)$.

$$\varphi_n(x) = \underbrace{\frac{(-1)^{n+1}}{(2n-1)^2}}_{|\cdots| = M_n} \sin(2n-1)x$$

Integral test

Aha! $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

The Fourier sine series for $f(x)$ <u>does</u> converge uniformly to some continuous fcn. Got to show that it is f!