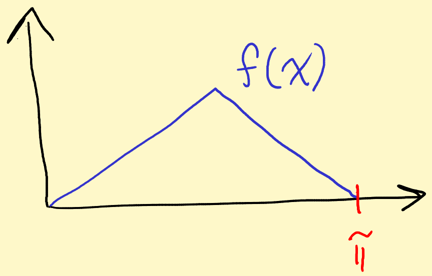
Lecture 4 Fourier sine series, full Fourier series

HWK 1 due in Gradescope 11:30pm Thurs. Late papers not accepted without prior permission



Fourier sine series for f
$$f(x) \text{ "=" } \sum_{n=1}^{\infty} A_n \sin nx$$
$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

Harpsichord solⁿ: $u(x,t) = \sum_{n=1}^{\infty} A_n \sin \underbrace{nx}_{\alpha} \cos \underbrace{nt}_{\beta}$

Trig identity $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

Johann's solⁿ $\Rightarrow$ Jacob's

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{2} A_n \left[\sin(\underbrace{nx+nt}_{n\underbrace{(x+t)}_{\theta}}) + \sin(\underbrace{nx-nt}_{n\underbrace{(x-t)}_{\psi}})\right]$$

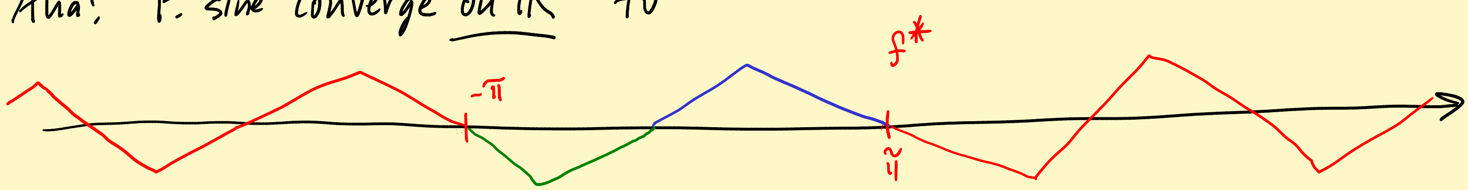
$$= \frac{1}{2}\left[\underbrace{\sum_{n=1}^{\infty} A_n \sin n\theta}_{f(\theta)} + \underbrace{\sum_{n=1}^{\infty} A_n \sin n\psi}_{f(\psi)}\right]$$

$$= \frac{1}{2}\left[f(x+t) + f(x-t)\right] \longleftarrow \text{Jacob's sol}^{n}!$$

Hmmm. What happens when $x+t$ and $x-t$ exit $[0,\pi]$?

Important observation about Fourier sine series $f(x) = \sum_{1}^{\infty} A_n \underbrace{\sin nx}_{\text{odd, } 2\pi \text{ periodic!}}$

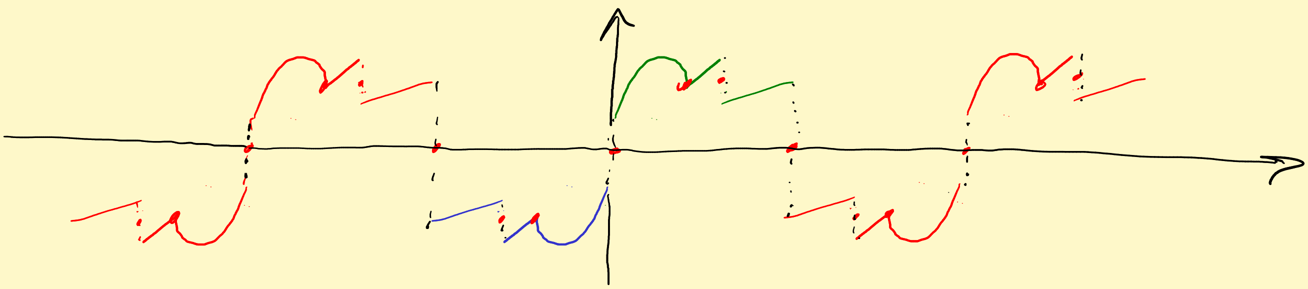
Aha! F. sine converge on $\mathbb{R}$ to

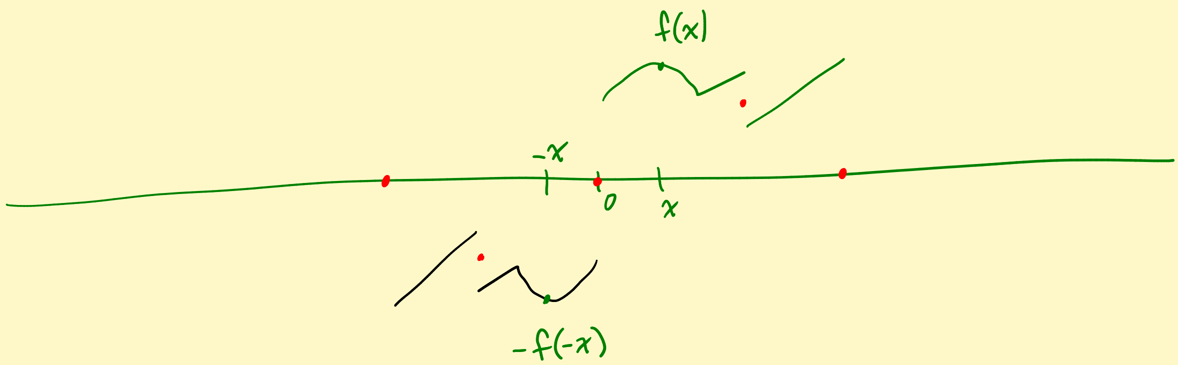


<u>Steps to get f^*</u> 1) Extend f from $[0,\pi]$ to $[-\pi,\pi]$ as an <u>odd</u> fcn.

2) Extend the fcn you get in (1) to $\mathbb{R}$ as a 2π-periodic fcn.

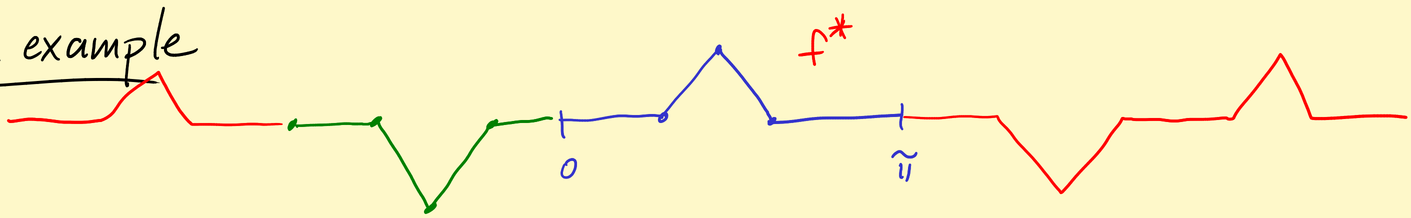
Correct version of Jacob's sol^n: $u(x,t) = \frac{1}{2}\left[\underbrace{f^*(x+t)}_{\text{moves left}} + \underbrace{f^*(x-t)}_{\text{moves right}}\right]$



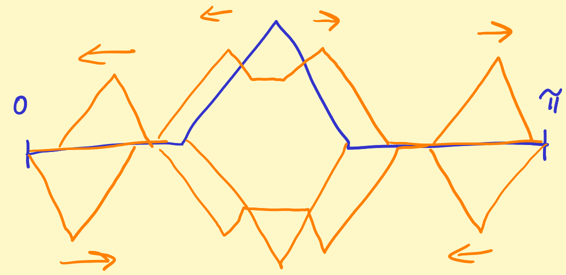


<u>Big fact</u> Fourier sine series converge to f^* on $\mathbb{R}$ (odd 2π-periodic extension of given f on $[0,\pi]$) using values $f(x)$ on $[0,\pi]$ where f is "smooth" and has "kinks", but we fill in the "jumps" of f^* we get giving values = midpoint of jump. [$\sin n\pi = 0$ for all $n \in \mathbb{Z}$, so F. Sine series $=0$ at $n\pi$'s.]

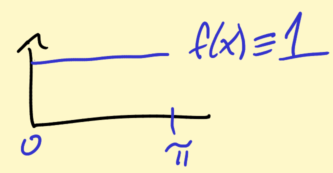
<u>Fun example</u>



Moving GIF
in AFTERMATH



EX Find the F. Sine series for



$$A_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx \, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2\pi}{n}\left[-\underbrace{\cos n\pi}_{(-1)^n} + \underbrace{\cos 0}_{1}\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \frac{4}{\pi}\left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \dots\right) = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

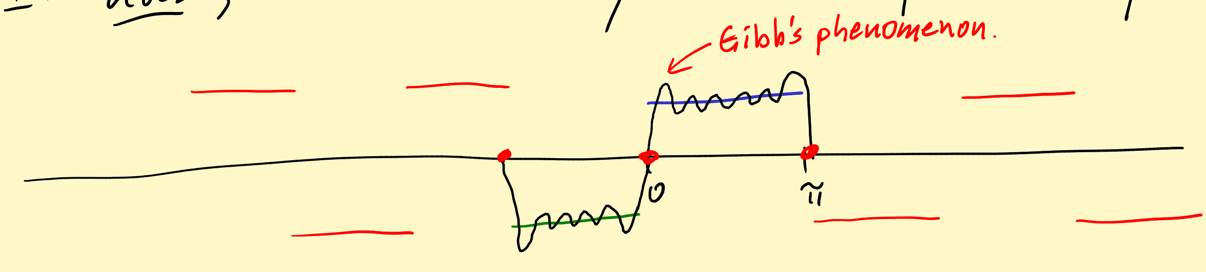
$\uparrow$ $n=1$ $\quad\uparrow$ 2 $\quad\uparrow$ 3 $\quad\dots$

$$f(x) \overset{?}{=} \frac{4}{\pi}\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

Ouch! $\Sigma|\text{Coeff}|$: $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ diverges to $+\infty$.

Can't use Weierstraß M-test to see F. series $\to$ <u>something</u>.

It <u>doess</u>, but more slowly than the "pluck" shape.



Full Fourier series $f(x) \text{ "=" } a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Key: $\{1, \cos nx, \sin nx : n=1,2,3,\ldots\}$ are $\perp$ on $[-\pi, \pi]$.

Check: $\displaystyle\int_{-\pi}^{\pi} 1 \cdot \underbrace{\cos nx}_{\text{or } \sin x}\, dx = 0 \qquad n=1,2,\ldots$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad \text{for all } n, m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi & n=m \\ 0 & n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi & n=m \\ 0 & n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} 1^2 \, dx = 2\pi$$

What do a_n's and b_n's have to be?

$$\int_{-\pi}^{\pi} f(x) \cos 3x \, dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right) \cos 3x \, dx$$

$$= 0 + \cdots + a_3 \underbrace{\int_{-\pi}^{\pi} \cos^2 3x \, dx}_{=\pi} + 0 + 0 + \cdots$$

So $$a_3 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 3x \, dx$$

Famous formulas:

$$\begin{cases} a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \, dx \quad \leftarrow \text{ave of } f \\ a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{cases}$$