Lecture 5 Complex version of Fourier series, even & odd functions

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \quad \text{where}$$

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \longleftarrow \text{ave of } f \text{ on } [-\pi,\pi] \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n \geq 1) \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{cases}$$

Complex version: $z = x + iy$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$= \underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}_{\cos y} + i\underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}_{\sin y}$$

$$e^{x+iy} \overset{\text{def}^n}{=} \left(e^x \cos y\right) + i\left(e^x \sin y\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\left(\text{Easy to check: } e^{-i\theta} = \frac{1}{e^{i\theta}}\right)$$

$$+ \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \underbrace{\cos nx}_{\frac{e^{inx}+e^{-inx}}{2}} + b_n \underbrace{\sin nx}_{\frac{e^{inx}-e^{-inx}}{2i}} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\underbrace{\left(\frac{a_n}{2} - i\frac{b_n}{2} \right)}_{c_n} e^{inx} + \underbrace{\left(\frac{a_n}{2} + i\frac{b_n}{2} \right)}_{c_{-n}} e^{-inx} \right]$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\left[\cos nx - i \sin nx \right]}_{e^{-inx}} dx$$

$$\boxed{c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx}$$

Let $c_0 = a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cdot \underset{\uparrow\, e^{-i \cdot 0 \cdot x}}{1}\, dx$

Great thing! c_n formula holds for c_{-n} by replacing n by $-n$.

One formula for all c_n's ! [Check it: Just write it out.]

Summary:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

Real inner product: $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

Integrals of <u>complex</u> valued functions of a <u>real</u> variable.

<u>EX</u> $F(t) = u(t) + i\,v(t)$

$$\int_{-\pi}^{\pi} F(t)\,dt = \lim_{|\Delta t| \to 0} \sum_{i=1}^{N} F(t_i)\Delta t_i = \lim_{|\Delta t| \to 0} \left[\sum_{i=1}^{N} u(t_i)\Delta t_i + i \sum_{i=1}^{N} v(t_i)\Delta t_i \right]$$

$$\int_{-\pi}^{\pi} F(t)\,dt \overset{\text{def}^{n}}{=} \int_{-\pi}^{\pi} u(t)\,dt + i \int_{-\pi}^{\pi} v(t)\,dt$$

<u>EX</u>: $$\int_{-\pi}^{\pi} e^{inx}\,dx = \int_{-\pi}^{\pi} \cos nx + i \sin nx \,dx$$

$$= \int_{-\pi}^{\pi} \cos nx\,dx + i \int_{-\pi}^{\pi} \sin nx\,dx$$

$$= 0 + i\,0 = 0$$

conjugate

Complex inner product: $\langle F, G \rangle = \int_{-\pi}^{\pi} F(t)\, \overline{G(t)}\, dt$

Big fact: $\{e^{inx}\}_{n=-\infty}^{\infty}$ are $\perp$ on $[-\pi,\pi]$.

Check: $\displaystyle\int_{-\pi}^{\pi} e^{inx}\,\overline{e^{imx}}\,dx$

$$\overline{e^{imx}} = \overline{\cos mx + i\sin mx} = \cos mx - i\sin mx$$
$$= \cos(-mx) + i\sin(-mx)$$

$$= \int_{-\pi}^{\pi} \underbrace{e^{inx}e^{-imx}}_{e^{inx-imx} = e^{i(n-m)x}}\,dx$$

$$= \int_{-\pi}^{\pi} \cos(n-m)x\,dx + i\int_{-\pi}^{\pi} \sin(n-m)x\,dx$$

$$= \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

Aha! $\displaystyle\int_{-\pi}^{\pi} f(x)\,e^{-imx}\,dx = \int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} c_n e^{inx}\right) e^{-imx}$ want

$$= \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx}_{= \begin{cases} 0 & n\neq m \\ 2\pi & n=m\end{cases}}$$

$$= c_m \cdot 2\pi$$

Fun facts

Back to real version of F.S.

$$f(x) = \underset{\uparrow \text{ even}}{a_0} + \sum_{n=1}^{\infty} \left(a_n \underbrace{\cos nx}_{\text{even}} + b_n \underbrace{\sin nx}_{\text{odd}} \right)$$

Fact Any fcn = (even fcn) + (odd fcn) ← unique decomp (when f cont at 0)

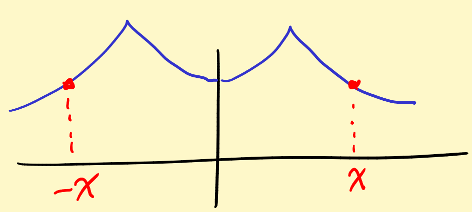
$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even!}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd!}}$$

Useful facts If is odd, all $\overbrace{a_n \text{ terms}}^{\text{cosine terms and } a_0 = 0} = 0$

If is even, all $\overbrace{b_n \text{ terms}}^{\text{sine terms} = 0} = 0$

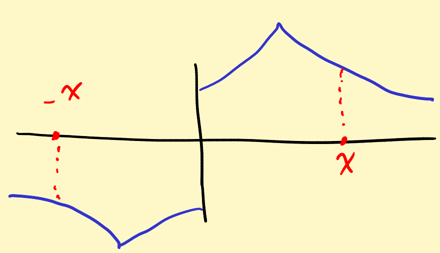
why: $$\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}} \; \underbrace{\cos nx}_{\text{even}}}_{\text{odd}} \, dx = 0$$ because $[-\pi, \pi]$ is symmetric about 0.

Even



$f(x)$ even:

$$\int_{-A}^{A} f(x)\,dx = 2\int_{0}^{A} f(x)\,dx$$



$f(x)$ odd:

$$\int_{-A}^{A} f(x)\,dx = 0$$

Simple facts

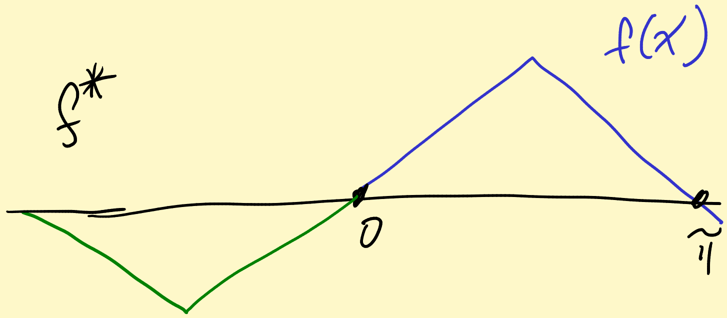
(even)(even) = even

(odd)(odd) = even

(even)(odd) = odd

Fourier sine and cosine series on $[0, \pi]$ pop out from full Fourier series on $[-\pi, \pi]$.

EX



extend as odd fcn.

Full Fourier series $f(x) = \cdots$

$= $ all cosine terms gone

$$= \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f^*(x)}_{\text{odd}} \underbrace{\sin nx}_{\text{odd}}}_{\text{even}}\, dx = \frac{1}{2\pi} \cdot 2 \int_0^{\pi} f(x) \sin nx\, dx$$