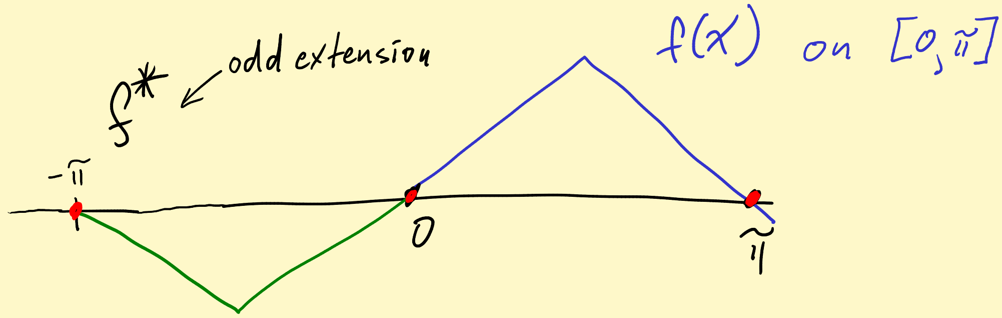
Lecture 6 The complex exponential function

Last time:

Fourier sine and cosine series on $[0,\pi]$ pop out from full Fourier series on $[-\pi,\pi]$.

Sine series:



extend as odd fcn.

Full Fourier series $f(x) = \cdots$

$= $ all cosine terms gone

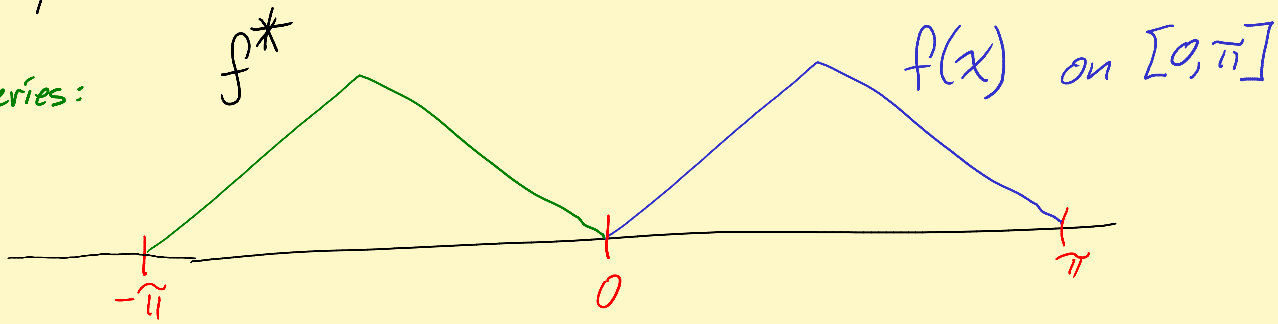
$$= \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f^*(x)}_{\text{odd}} \underbrace{\sin nx}_{\text{odd}}}_{\text{even}} \, dx = \underbrace{\frac{1}{\pi}\cdot 2}_{\frac{2}{\pi}} \int_0^{\pi} f(x) \sin nx \, dx$$

Fourier sine series b_n ✓

Similarly

Cosine series:



extend f from $[0,\pi]$ to $[-\pi,\pi]$ as an even fcn.

Full Fourier series for f^* on $[-\pi, \pi]$: All b_n sine terms $= 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f^*(x)}_{\text{even}} \underbrace{\cos nx}_{\text{even}}}_{\text{even}} \, dx = 2 \cdot \frac{1}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

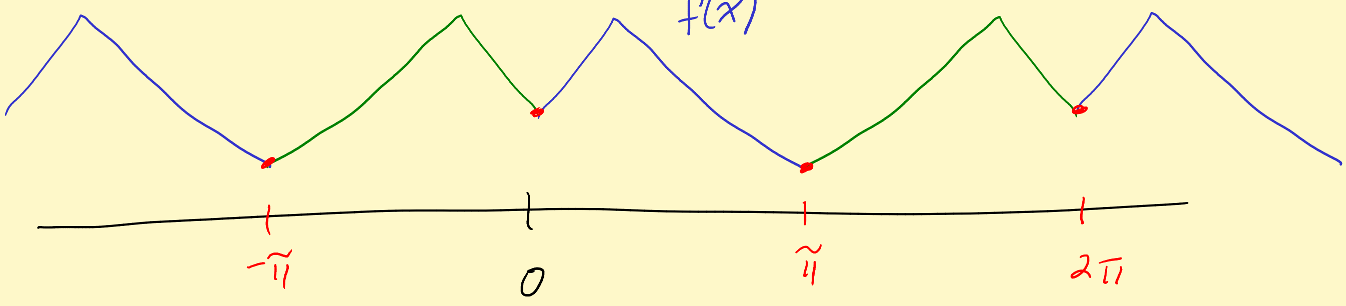
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x)\, dx = 2 \cdot \frac{1}{2\pi}\int_0^{\pi} f(x)\, dx$$

Get Fourier cosine series on $[0, \pi]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{cases} a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx \leftarrow \text{ave of } f \text{ on } [0,\pi] \\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx, \quad n \geq 1. \end{cases}$$

Remark Sine series converge to 0 at 0, $n\pi$ $n \in \mathbb{Z}$, which are midpoints of jumps of "odd 2π-periodic extension of f" to $\mathbb{R}$. Cosine series doesn't have jumps at $n\pi$'s



even 2π-periodic extension of $f(x)$ hook up at $n\pi$'s.

Complex exponential $z = x + iy$

Today only $E(z) = e^z$

$$E(x+iy) = (e^x \cos y) + i(e^x \sin y)$$

Properties

0) $E(0) = E(0+0i) = 1$

1) $E(z+w) = E(z)E(w) \leftarrow$ Group homomorphism: (Additive group on $\mathbb{C}$) $\rightarrow$ (Multiplicative group on $\mathbb{C} - \{0\}$.)

2) $1 = E(0) = E(z-z) = \underbrace{E(z)}_{}\, E(-z)$

$E(z)$ is <u>never</u> zero.

$$E(-z) = 1/E(z)$$

3) $E(z-w) = E(z)/E(w)$

4) $E(x) = E(x+0i) = e^x(1+0i) = e^x$

$E(z)$ extends e^x to $\mathbb{C}$.

5) $E(iy) = \cos y + i \sin y$

Proof of (1)

$$E(z_1)E(z_2) = E(x_1+iy_1)E(x_2+iy_2)$$

$$= e^{x_1}[\cos y_1 + i \sin y_1]\, e^{x_2}[\cos y_2 + i \sin y_2]$$

$$= \underbrace{\left(e^{x_1} e^{x_2}\right)}_{e^{x_1+x_2}} \left[\underbrace{\left(\cos y_1 \cos y_2 \overset{i^2=-1}{-} \sin y_1 \sin y_2\right)}_{\cos(y_1+y_2)} + i\left(\underbrace{\cos y_1 \sin y_2 + \cos y_2 \sin y_1}_{\sin(y_1+y_2)}\right)\right]$$

$$= E\left((x_1+x_2) + i\,(y_1+y_2) \right) \checkmark$$

Remark Easy to remember sum of angles formulas!

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

(Euler identity, multiply, equate real & imag parts.)

DeMoivre's formula $\left(e^{i\theta}\right)^N = e^{iN\theta} = \cos N\theta + i \sin N\theta$

for $N \in \mathbb{Z}$.

Important tool for finding all the N-th roots of a complex #.

Pf: $N \in \mathbb{N}$: $\left(e^{i\theta}\right)^N = \underbrace{e^{i\theta} e^{i\theta} \cdots e^{i\theta}}_{N\text{-times}} = e^{i(\underbrace{\theta+\theta+\cdots+\theta}_{N\text{-times}})}$

$$= e^{iN\theta} \checkmark$$

Negative N. Apply $\left(e^{-i\theta}\right) = \frac{1}{e^{i\theta}}$ to above.

Fun fact $E(z)$ is complex differentiable, meaning that

$$\lim_{z\to a} \frac{E(z)-E(a)}{z-a}$$ exists. Call limit $E'(a)$.

Big property : $\left.\begin{array}{l} E'(z) = E(z) \\ E(0) = 1 \end{array}\right\} \leftarrow$ these determine $E(z)$ uniquely

EX: DQ for z^2 : $DQ = \frac{z^2-a^2}{z-a} = \frac{(z-a)(z+a)}{z-a} = z+a \to 2a$ as $z \to a$.

Important function: Dirichlet kernel

$$D_N(t) = \sum_{n=-N}^{N} e^{int}$$

$$= \sum_{n=1}^{N} e^{-int} + \sum_{n=0}^{N} e^{int}$$

$$= \sum_{n=1}^{N} (\cos nt - i\sin nt) + \underset{\substack{\uparrow \\ n=0}}{1} + \sum_{n=1}^{N} (\cos nt + i\sin nt)$$

$$= 1 + 2\sum_{n=1}^{N} \cos nt = \frac{\sin[(N+\frac{1}{2})t]}{\sin\frac{t}{2}}$$

<u>Why</u> $1 + \cos t + \cos 2t + \cdots + \cos Nt \leftarrow (*)$

$$= \operatorname{Re}\Big[\underbrace{1 + e^{it} + e^{i2t} + \cdots + e^{iNt}}_{1 + (e^{it}) + (e^{it})^2 + \cdots + (e^{it})^N}\Big]$$

Geom series!

<u>$z = e^{it}$</u>

$$S_N = 1 + z + z^2 + \cdots + z^N$$

$$- \quad zS_N = \quad z + z^2 + \cdots + z^N + z^{N+1}$$

$$(1-z)S_N = 1 - z^{N+1}$$

$$\boxed{S_N(z) = \frac{1-z^{N+1}}{1-z}}$$

$$(*) = Re\left[\frac{1-(e^{it})^{N+1}}{1-e^{it}}\right]$$

$$= Re\left[\frac{1-e^{i(N+1)t}}{1-e^{it}} \cdot \frac{e^{-it/2}}{e^{-it/2}} \cdot \frac{\left(\frac{-1}{2i}\right)}{\left(\frac{-1}{2i}\right)}\right]$$

$$= Re\left[\frac{\left(\frac{i}{2}e^{-it/2} - \frac{i}{2}e^{i(N+\frac{1}{2})t}\right)}{\left(\frac{e^{it/2}-e^{-it/2}}{2i}\right)}\right] \leftarrow \left(\frac{1}{i} = -i\right)$$

$\leftarrow \sin\frac{t}{2}$

$$= \frac{\frac{1}{2}\sin\frac{t}{2} + \frac{1}{2}\sin\left(N+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

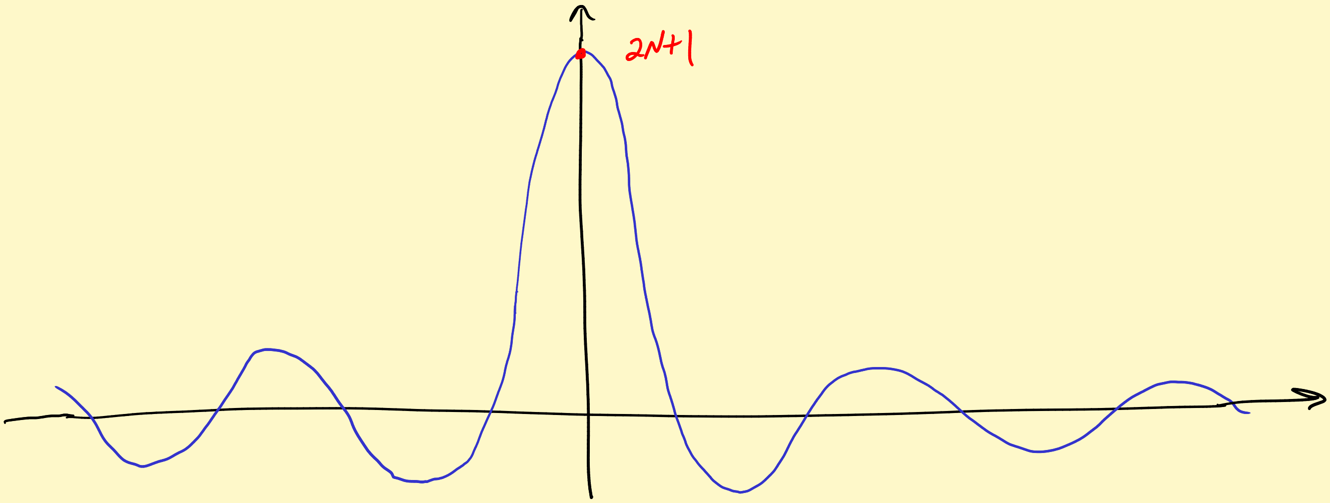
$$= \frac{1}{2} + \frac{1}{2}\,\frac{\sin\left(N+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

Wow!

$$D_N(t) = 1 + 2\sum_{n=1}^{N}\cos nt$$

$$= 2(*) - 1 = \frac{\sin\left(N+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \checkmark$$

L'H shows $D_N(t) \xrightarrow[t\to 0]{} \dfrac{(N+\frac{1}{2})\cos 0}{\frac{1}{2}\cos 0} = 2N+1$ Big!



One key property: $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} D_N(t)\,dt = 1$

(easy because $\underbrace{\displaystyle\int_{-\pi}^{\pi} \cos nt\,dt}_{=0}$ in sum.)

Words $f(x) \cong \underbrace{a_0 + \displaystyle\sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)}_{\text{called a trigonometric poly}}$

$$= \underbrace{\sum_{n=-N}^{N} c_n e^{inx}}$$

also called a trigonometric poly: $\displaystyle\sum_{-N}^{N} c_n e^{inx}$

complex #s

Read Stein Chap 2.