Lecture 7 Bessel's inequality

Hwk 2 due in GS Thurs, 11:30 pm

Real inner product: $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$ ← f, g real valued

Complex inner product: $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$ ← conjugate

(f, g allowed to be complex valued)

Bessel's inequality Assume $\varphi_n(x)$ are continuous real valued fcns on $[a,b]$ that are orthonormal, meaning that

$$\langle \varphi_n,\varphi_m\rangle = 0 \quad \text{when } n\neq m \quad (\text{orthogonal})$$

$$\langle \varphi_n,\varphi_n\rangle = \int_a^b \varphi_n(x)^2\,dx = 1 \quad \leftarrow \text{"normalized"}$$

Def^n The L^2-norm of φ is $\sqrt{\langle\varphi,\varphi\rangle} = \sqrt{\int_a^b \varphi(x)^2\,dx}$

Remark Can always "normalize" an orthogonal family of fcns [such as $\{\sin nx\}_{n=1}^{\infty}$ on $[0,\pi]$] by dividing them by their L^2-norms. $\underset{\text{orthonormal}}{\tilde{\varphi}_n} = \frac{1}{c_n}\underset{\text{orthogonal}}{\varphi_n}$ where $c_n = \sqrt{\langle\varphi_n,\varphi_n\rangle}$.

Hope: $f = \sum_{n=1}^{\infty} c_n\varphi_n$ (like Fourier sine series)

Then: $\langle f,\varphi_m\rangle = \sum_{n=1}^{\infty} c_n \underbrace{\langle\varphi_n,\varphi_m\rangle}_{\text{all zero if } n\neq m}$

$$= c_m \underbrace{\langle \varphi_m, \varphi_m \rangle}_{=1}$$

If $f = \sum_{n=1}^{\infty} c_n \varphi_n$, then c_n has to be $\boxed{c_n = \langle f, \varphi_n \rangle}$

Today: Define $c_n = \langle f, \varphi_n \rangle$.

Big fact $\sum_{n=1}^{N} c_n \varphi_n$ is the best L^2 approximation to f on $[a,b]$ among fcns of the form

$\sum_{n=1}^{N} A_n \varphi_n$, meaning

$$\left\| f - \sum_{n=1}^{N} c_n \varphi_n \right\| \le \left\| f - \sum_{n=1}^{N} A_n \varphi_n \right\|$$

with equality only when $A_n = c_n$ for $n = 1, 2, \ldots, N$.

Furthermore $\sum_{n=1}^{N} c_n^2 \le \|f\|^2$ ← Bessel's ineq

Aha! $\sum_{n=1}^{\infty} c_n^2$ converges to limit $\le \|f\|^2$.

Cor Fourier coeff of a continuous fcn f on $[-\pi, \pi]$ tend to zero as $n \to \infty$. $\begin{cases} a_n \to 0 \\ b_n \to 0 \end{cases}$

(So do complex versions $c_n \to 0$ as $n \to +\infty$, $n \to -\infty$

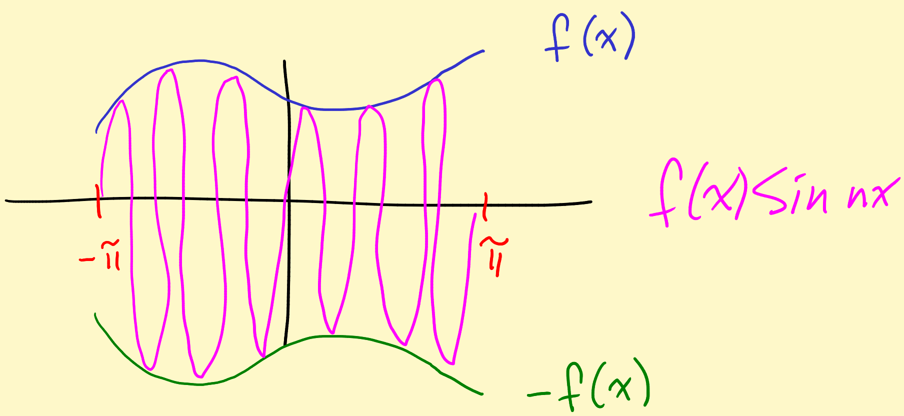
because they are simple sums of a_n's and b_n's.)

Interesting! Not obvious because

$$|b_n| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right|$$

$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|f(x)|}_{\le M}\;\underbrace{|\sin nx|}_{\le 1}\,dx$$

$$\le \frac{1}{\pi}(M \cdot 1)(2\pi) = 2M$$ ← Ouch! Does not → 0



Aha! Looks like area under positive humps cancels area under negative humps!

Pf

$$E = E^* = \left\|f - \sum_{1}^{N} \overset{A_n}{c_n}\varphi_n\right\|^2$$

$$= \int_a^b \underbrace{\left(f - \sum_{n=1}^{N} \overset{\overset{A_n}{\downarrow}}{c_n} \varphi_n\right)\left(f - \sum_{m=1}^{N} \overset{\overset{A_m}{\downarrow}}{c_m} \varphi_m\right)}_{=\text{ square}} dx$$

$$= \int_a^b f^2\, dx - \sum_{n=1}^{N} \overset{\overset{A_n}{\downarrow}}{c_n} \underbrace{\int_a^b f \varphi_n\, dx}_{=c_n} - \sum_{n=1}^{N} \overset{\overset{A_m}{\downarrow}}{c_m} \underbrace{\int_a^b f \varphi_m\, dx}_{=c_m}$$

$$+ \sum_{n=1}^{N} \sum_{m=1}^{N} \overset{\overset{A_n}{\downarrow}}{c_n} \overset{\overset{A_m}{\downarrow}}{c_m} \underbrace{\int_a^b \varphi_n \varphi_m\, dx}_{\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}}$$

Let $S' = \sum_{n=1}^{N} c_n^2$

Got $\underbrace{\left\| f - \sum_{1}^{N} c_n \varphi_n \right\|^2}_{\geq 0} = \int_a^b f^2\, dx - S' - S' + S'$

$$= \int_a^b f^2\, dx - \sum_{n=1}^{N} c_n^2 \geq 0$$

Bessel's ineq : $\sum_{n=1}^{N} c_n^2 \leq \int_a^b f^2\, dx$ ✓

$$\begin{cases} E = \|f\|^2 - 2\sum_{n=1}^{N} A_n c_n + \sum_{n=1}^{N} A_n^2 \\ E^* = \|f\|^2 - \sum_{n=1}^{N} c_n^2 \end{cases}$$

Aha! $E - \underset{\substack{\uparrow \\ \text{best}}}{E^*} = \sum_{n=1}^{N} \Big(\underbrace{c_n^2 - 2A_n c_n + A_n^2}_{(c_n - A_n)^2} \Big)$

Sum of squares is ≥ 0 !

So $E - E^* \geq 0$. $\boxed{E^* \leq E}$ $\leftarrow$ worse if $<$ in ineq

What if $E = E^*$? Sum of squares $= 0$ only happens if <u>each</u> term $= 0$: $(c_n - A_n)^2 = 0$ each n.

$\Rightarrow$ $c_n = A_n$ for $n = 1, \dots, N$ ✓

<u>EX</u> $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, n = 1, 2, 3, \dots \right\}$

are <u>orthonormal</u>. $\begin{cases} \varphi_1(x) = \frac{1}{\sqrt{2\pi}} \\ \varphi_{2n}(x) = \frac{1}{\sqrt{\pi}} \cos nx \\ \varphi_{2n+1}(x) = \frac{1}{\sqrt{\pi}} \sin nx \end{cases}$ in Bessel's

$$\boxed{2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx}$$

Bessel's ineq for Fourier series.

Big fact For Fourier series, Bessel's ineq turns out to be an equality: Parseval's identity.

Reason: Fourier functions form a complete orthonormal basis for $L^2[-\pi,\pi]$ ← Hilbert space of square integrable fcns on $[-\pi,\pi]$.

Can think: $f \longmapsto \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$ ← act just like coordinates.

$$\|f\|^2 = \int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{1}^{\infty} c_n^2$$

$$L^2[-\pi,\pi] \approx \ell^2$$ ← square sumable sequences

Hilbert space isomorphism