Lecture 8 Fourier series converge! (sometimes)

HWk 3 due in two weeks

Last time: Bessel's ineq using $\varphi_1(x) = \frac{1}{\sqrt{2\pi}}$, $\varphi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}$, $\varphi_{2n+1}(x) = \frac{\cos nx}{\sqrt{\pi}}$

on $[-\pi, \pi]$ gives us

$$2a_0^2 + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

[so we can let $N \to \infty$ in this ineq.]

for piecewise continuous fcns f on $[-\pi, \pi]$

Cor Fourier coeff of a piecewise continuous fcn on $[-\pi, \pi]$ $\to 0$ as $n \to \infty$. (for real and complex versions)

Theorem Assume f is C^1-smooth on $\mathbb{R}$ and 2π-periodic.

Then the Fourier series

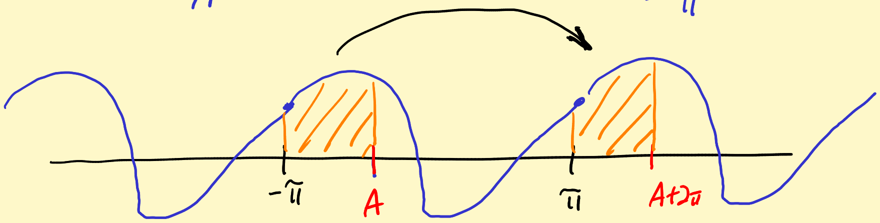
$$\underbrace{a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=-N}^{N} c_n e^{inx}}_{S_N(x)}$$

converges to $f(x)$ at each $x \in \mathbb{R}$.

Lemma If f is 2π-periodic, then

$$\int_A^{A+2\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx$$

Pf



Pf of thm

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

$$= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt \right) e^{inx}$$

$$= \int_{-\pi}^{\pi} \underset{!}{\underline{\underline{\left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-t)} \right)}}} f(t)\, dt$$

Dirichlet kernel! $D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}$

The 2π is new here today.

Fantastic formula: $S_N(x) = \int_{-\pi}^{\pi} D_N(x-t)\, f(t)\, dt$

Facts

1) $\int_{-\pi}^{\pi} D_N(t)\, dt = \underline{1}$

2) $D_N(t) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin \frac{t}{2}}$ $\quad t \neq 0$

$D_N(0) = \frac{1}{2\pi} \frac{(N+\frac{1}{2})}{(\frac{1}{2})} = \frac{N+\frac{1}{2}}{\pi}$ $\quad t = 0$

3) $D_N(t)$ is 2π-periodic.

Pf of 1 : $$\int_{-\pi}^{\pi} D_N(t)\,dt = \underbrace{\int_{-\pi}^{\pi} \frac{1}{2\pi}\cdot 1\,dt}_{n=0} + \sum_{\substack{|n|\le N\\ n\neq 0}} \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} e^{int}\,dt}_{=0}$$

$e^{int} =$ $\cos nt + i\sin nt$

$$= 1 + \sum 0's$$

$$= 1$$

Pf of 3 : All terms in $D_N(t)$ are 2π-periodic.

Bad things about $D_N(t)$ 1) $D_N(t)$ is not ≥ 0 on $[-\pi,\pi]$.

2) $\int_{-\pi}^{\pi} |D_N(t)|\,dt \longrightarrow \infty$ as $N \to \infty$.

Pf cont'd $$S_N(x) = \int_{-\pi}^{\pi} D_N(\underbrace{x-t}_{\tau})\, f(t)\,dt$$

$\tau = x - t$ $\boxed{t = x - \tau}$

$d\tau = -dt$

When $t = -\pi$: $\tau = x + \pi$

$t = \pi$: $\tau = x - \pi$

$$= \int_{\tau = x+\pi}^{x-\pi} D_N(\tau) f(x-\tau)\,(-d\tau)$$

$$= \int_{x-\pi}^{x+\pi} D_N(\tau) f(x-\tau)\,d\tau$$

Change dummy var τ back to t. Get

Lemma $\Longrightarrow$ $$\boxed{S_N(x) = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\,dt}$$

Hmmm. $\underline{f(x) = 1 \cdot f(x)} = \underbrace{\left(\int_{-\pi}^{\pi} D_N(t)\,dt \right)}_{=1} \cdot f(x)$

$$\boxed{f(x) = \int_{-\pi}^{\pi} D_N(t)\, f(x)\, dt}$$

Finally $$\boxed{S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t) \left[f(x-t) - f(x) \right] dt}$$

$$(*) = \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}} \left[f(x-t) - f(x) \right] dt$$

$$\sin(N+\tfrac{1}{2})t = \sin Nt \cos\tfrac{t}{2} + \cos Nt \sin\tfrac{t}{2}$$

$$(*) = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin Nt \underbrace{\left[\frac{1}{2} \cos\frac{t}{2} \underbrace{\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}}_{g(t)} \right]}_{h(t)} dt$$

↖ b_N for h

$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} \cos Nt \underbrace{\left[\frac{1}{2} \sin\frac{t}{2} \underbrace{\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}}_{g} \right]}_{\tilde{h}(t)} dt$$

↖ a_N for $\tilde{h}$

Aha! These are the N-th Fourier coeff a_N for $\tilde{h}$, b_N for h,

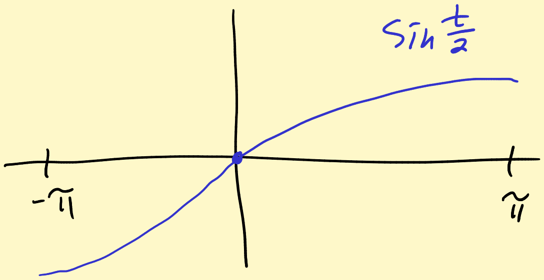
$h, \tilde{h}$ nice continuous fcns on $[-\pi, \pi]$. They $\to 0$ as $N \to \infty$.

So $S_N(x) \to f(x)$ as $N \to \infty$. ✓

Why they are "nice":

$$g(t) = \frac{f(x-t)-f(x)}{\sin\frac{t}{2}} \begin{array}{l} \leftarrow =0 \text{ at } t=0 \\ \leftarrow =0 \text{ at } t=0 \end{array}$$

$$\text{L'H} \quad \lim_{t \to 0} g(t) = \lim_{t \to 0} \frac{-f'(x-t)}{\frac{1}{2}\cos\frac{t}{2}} = -2f'(x)$$



See $g(t)$ is continuous on $[-\pi, \pi]$.

$h(t) = \frac{1}{2}\cos\frac{t}{2}\, g(t)$ nice ✓

$\tilde{h}(t) = \frac{1}{2}\sin\frac{t}{2}\, g(t)$ nice ✓

Bessel's ineq $\Longrightarrow$ $S_N(x) \to f(x)$ for each x.

See "pointwise" convergence.

But how fast? Feeling: the smoother the fcn, the faster the convergence.

Next time f is $\mathcal{C}^2$-smooth, 2π-periodic,

then $\left| s_N(x) - f(x) \right| < \frac{M_2}{N}$ for

some constant M for all x. Unif convergence!

... $\mathcal{C}^3$-smooth .... $< \frac{M_3}{N^2}$ ...

... $\mathcal{C}^4$-smooth ... $< \frac{M_4}{N^3}$ ...

... $\mathcal{C}^K$-smooth ... $< \frac{M_K}{N^{K-1}}$ ...

History Bessel's ineq $\Longrightarrow$ $s_N \to f$ in L^2-norm!

This is where we see Riemann integral has issues.

Need Lebesgue integral!