Lecture 9 More about the convergence of Fourier series

$$S_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx \right) = \sum_{n=-N}^{N} c_n e^{inx} =$$

$$= \int_{-\pi}^{\pi} D_N(x-t) f(t)\, dt = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\, dt$$

extend f from $[-\pi, \pi]$ to $\mathbb{R}$ as a 2π-periodic fcn

$$f(x) = f(x) \underbrace{\int_{-\pi}^{\pi} D_N(t)\, dt}_{=1} = \int_{-\pi}^{\pi} D_N(t)\, f(x)\, dt$$

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t) \left[f(x-t) - f(x) \right] dt$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin (N+\frac{1}{2})t}{\sin t/2} \left[f(x-t) - f(x) \right] dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin Nt \underbrace{\left[\frac{1}{2} \cos \frac{t}{2} \underbrace{\frac{f(x-t) - f(x)}{\sin \frac{t}{2}}}_{g(t)} \right]}_{h(t)} dt$$

$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} \cos Nt \underbrace{\left[\frac{1}{2} \sin \frac{t}{2} \underbrace{\frac{f(x-t) - f(x)}{\sin t/2}}_{g(t)} \right]}_{\tilde{h}(t)} dt$$

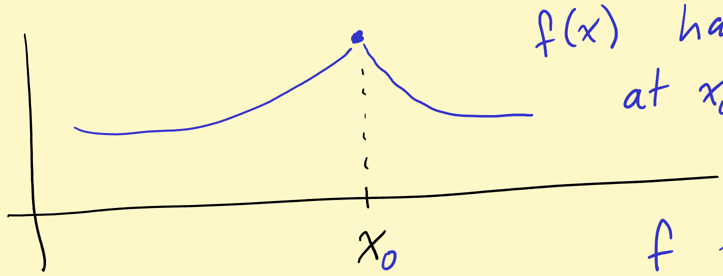
First $\int_{-\pi}^{\pi}$ is b_N coeff for h. Second is a_N coeff for $\tilde{h}$.

Bessel's ineq $\Rightarrow$ these go to zero whenever g is piecewise continuous. ← It can have jumps.

Fact f C^1-smooth and 2π-periodic on $\mathbb{R}$

$\Rightarrow$ g is continuous. So Fourier series converges to f pointwise in this case.

What if the graph of f has a kink?



$f(x)$ has right hand derivative at x_0 and a left hand one. Perhaps not equal.

f piecewise C^1-smooth

How nice is $g(t) = \dfrac{f(x-t)-f(x)}{\sin t/2}$?

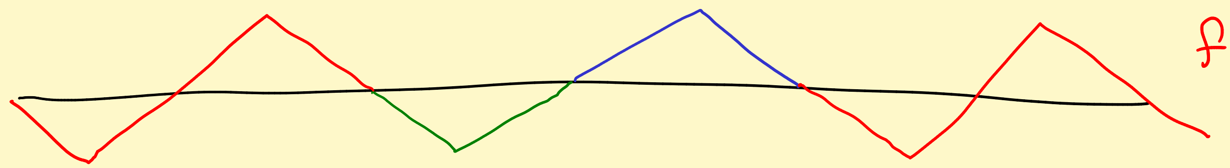
Can do L'H from left and right.

$$L'H_{\pm} \begin{cases} \lim\limits_{t\to 0+} \dfrac{f(x-t)-f(x)}{\sin t/2} = \dfrac{-f_L'(x_0)}{\left(\frac{1}{2}\right)} \\[2ex] \lim\limits_{t\to 0-} \dfrac{f(x-t)-f(x)}{\sin t/2} = \dfrac{-f_R'(x_0)}{\left(\frac{1}{2}\right)} \end{cases}$$

different at kink

g is continuous where f is C^1. At kinks, g has a jump. That's ok! Bessel's still holds.

Thm f is piecewise C^1-smooth and continuous and 2π-periodic (like the harpsichord pluck shape). Then the Fourier series converges to f pointwise.



$$f(x) = \frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots$$

p-series with $p=2$.

Hmmm. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} < \sum_{k=1}^{(2n+1)} \frac{1}{k^2} \to \frac{\pi^2}{6}$

integral test $\Rightarrow < \infty$.

Weierstraß M-test $\Rightarrow$ Series converges uniformly to a continuous fcn. Today we know that fcn is $f(x)$!

New today If f is C^2-smooth and 2π-periodic on $\mathbb{R}$, then series converges uniformly to f because

$$|a_n| < \frac{M}{n^2} \quad \text{and} \quad |b_n| < \frac{M}{n^2} \quad \text{and we can use}$$

Weir M-test as in harpsichord prob.

Why Integration by parts will show:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Bessel's $\Rightarrow c_n \to 0$

$$f'(x) \sim \sum_{n=-\infty}^{\infty} in\, c_n e^{inx}$$

$$f''(x) \sim \sum_{n=-\infty}^{\infty} (in)^2 c_n e^{inx}$$

Bessel's $-n^2 c_n \to 0$

<u>Note</u> $n^2 c_n \to 0$. Let $\varepsilon = 1$. There is an $N \in \mathbb{N}$ such that $|n^2 c_n| < 1$ when $n \geq N$.

So $|c_n| < \frac{1}{n^2}$ when $n \geq N$.

<u>Notation from Stein</u> $\hat{f}(n) = c_n$ for f.

<u>Pf</u> that $\widehat{f'}(n) = in\, \hat{f}(n)$

$$\widehat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{-inx}}_{u} \quad \underbrace{f'(x)\, dx}_{dv}$$

$du = -in e^{-inx} dx \qquad v = f(x)$

$$= \frac{1}{2\pi}\left[uv\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v\, du \right]$$

$$= \frac{1}{2\pi}\left[e^{-inx} \cdot f(x) \right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \left[-in e^{-inx}\, dx \right]$$

$$= \quad 0 \quad + \quad in\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx\right)$$

fcns are 2π-periodic

$$= \quad in\,\hat{f}(n) \quad \checkmark$$

Do it again! $\quad \widehat{f''}(n) = (in)^2\,\hat{f}(n) \quad \checkmark$

Today's theorem: If f is C^2-smooth and 2π-periodic, then it is C^1-smooth too, so F.S. converges pointwise to f. Weierstraß M-test showed today that F.S. converges uniformly to a continuous fcn, which must be f!

Integration by parts also shows

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$f'(x) \sim 0 + \sum_{n=1}^{\infty}\left(-n a_n \sin nx + n b_n \cos nx\right)$$

$$f''(x) \sim \text{etc.}$$

Remark If f if Lipschitz continuous and 2π periodic on $\mathbb{R}$, the F.S. $\to f$ pointwise.

Lipshitz: $\quad |f(x) - f(y)| \leq C\,|x-y| \quad$ for $x, y \in \mathbb{R}$.

Pf: $g(t)$ is bounded near $t=0$, continuous elsewhere.

$$g(t) = \frac{f(x-t) - f(x)}{\sin \frac{t}{2}}$$

Riemann-Lebesgue lemma

Riemann: f Riemann integrable, then Fourier coeff $\to 0$ as $n \to \infty$.

Lebesgue: " Lebesgue integrable " "

We proved this when f is piecewise continuous using Bessel's ineq.