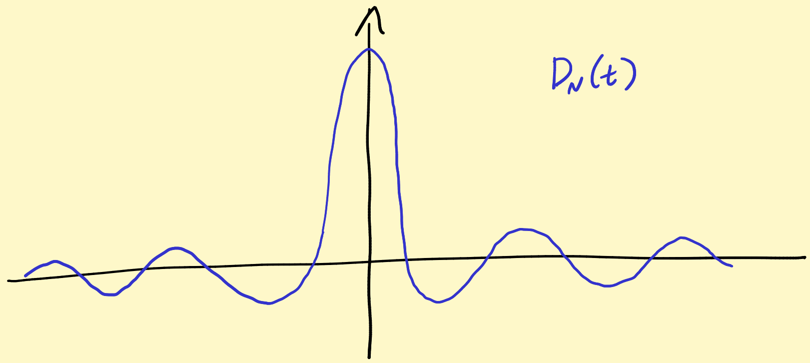
Lecture 10 The Fejér kernel and Fejér's theorem

The Dirichlet kernel $D_N(t) = \frac{1}{2\pi} \frac{\sin[(N+\frac{1}{2})t]}{\sin t/2}$

A wonderful kernel, but not a "good kernel"!



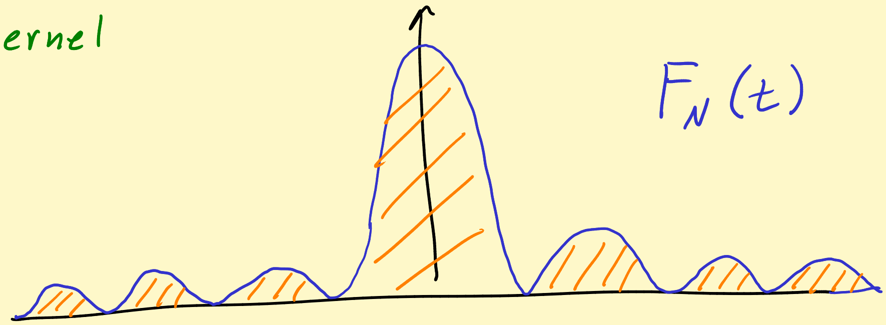
Good things: $\int_{-\pi}^{\pi} D_N(t)\,dt = 1$

Big at origin.

But $\int_{-\pi}^{\pi} |D_N(t)|\,dt \longrightarrow \infty$ as $N \to \infty$.

$D_N(t) \not\geq 0$ for all t.

Fejér kernel



Good things: $\int_{-\pi}^{\pi} F_N(t)\,dt = 1$

F_N big near 0,

$F_N(t) \geq 0$ for all t

$F_N(t) \to 0$ uniformly outside $[-\delta, \delta]$.

Important example: Stein: A continuous 2π-periodic fcn whose Fourier series does not converge at a pt x_0.

Big idea $\quad S_N(t) = \sum_{-N}^{N} c_n e^{inx}, \quad c_n = \hat{f}(n)$.

Assume f is continuous, 2π-periodic.

Consider the average of first N partial sums.

$$\sigma_N(x) = \frac{S_0(x) + S_1(x) + \cdots + S_{N-1}(x)}{N}$$

$$= \int_{-\pi}^{\pi} \underbrace{\left[\frac{D_0(t) + \cdots + D_{N-1}(t)}{N} \right]}_{F_N(t)} f(x-t)\, dt$$

Hmmm. Combining geometric series in e^{it}, e^{-it} and being clever in the algebra ...

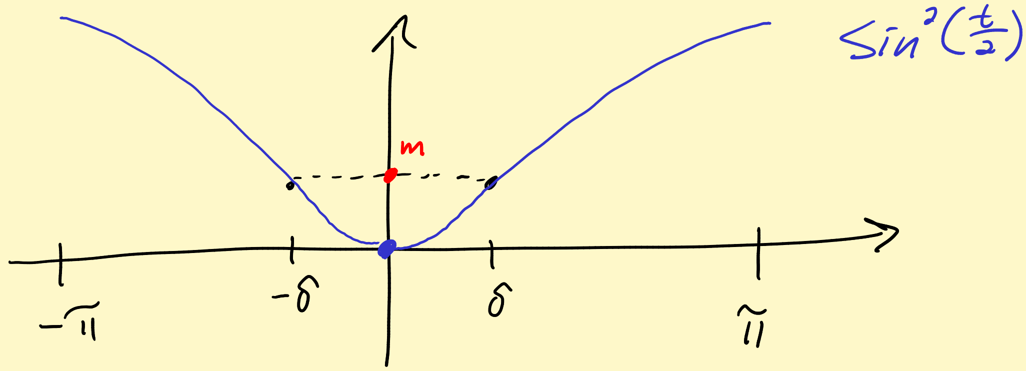
Get $$\boxed{F_N(t) = \frac{1}{2\pi N} \frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}}$$

Key properties 1) $F_N(t) \geq 0$ for all t. ✓

2) $\int_{-\pi}^{\pi} F_N(t)\,dt = 1$. Pf $D_N(t)$'s do that!

3) $\lim_{t \to 0} F_N(t) = \frac{N}{2\pi}$ Pf: L'H

4) Outside $(-\delta, \delta)$, $0 < \delta < \pi$, $F_N(t) \to 0$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$ as $N \to \infty$.



$$|F_N(t)| = \frac{1}{2\pi N} \left| \frac{\sin^2(N\frac{t}{2})}{\sin^2\frac{t}{2}} \right| \le \frac{1}{2\pi N} \cdot \frac{1}{m} \xrightarrow[N \to \infty]{} 0 \text{ as}$$

when t outside $(-\delta, \delta)$.

"Good kernel" satisfies 1-4.

Fejér's theorem If f is continuous and 2π-periodic, then $\sigma_N(x) \longrightarrow f(x)$ uniformly on $\mathbb{R}$!

Pf

$$\sigma_N(x) - f(x) = \int_{-\pi}^{\pi} F_N(t)\, f(x-t)\,dt - \underbrace{\int_{-\pi}^{\pi} F_N(t)\,dt}_{=1} \cdot f(x)$$

$$= \int_{-\pi}^{\pi} F_N(t)\,\left[f(x-t) - f(x) \right] dt$$

$$= \underbrace{\int_{-\delta}^{\delta} F_N(t) \left[\underbrace{f(x-t) - f(x)}_{\text{small}} \right] dt}_{I_1} + \underbrace{\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) \underbrace{F_N(t)}_{\text{small}} \left[f(x-t) - f(x) \right] dt}_{I_2}$$

Let $\varepsilon > 0$. f cont on $[-\pi, \pi] \Rightarrow f$ unif cont. on $[-\pi, \pi]$

$\Rightarrow$ unif cont on $\mathbb{R}$.

2π-periodic ✓

So $\exists\, \delta > 0$ such that $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$ when $|t| < \delta$

(independent of x).

$$|I_1| \leq \int_{-\delta}^{\delta} \underbrace{|F_N(t)|}_{= F_N(t)} \underbrace{|f(x-t) - f(x)|}_{< \frac{\varepsilon}{2}} dt$$

$$\leq \frac{\varepsilon}{2} \underbrace{\int_{-\delta}^{\delta} F_N(t)\, dt}_{\leq \underbrace{\int_{-\pi}^{\pi} F_N(t)\, dt}_{=1}}$$

because $F_N(t) \geq 0$

$|I_1| \leq \frac{\varepsilon}{2}$ ✓

Note f cont on $[-\pi,\pi] \Rightarrow f$ bounded there.

Say $|f| \le M$ on $[-\pi,\pi]$. 2π-periodic $\Rightarrow$

$|f| \le M$ on $\mathbb{R}$. So $|f(x-t)-f(x)| \le |f(x-t)| + |f(x)|$

$$\le \underbrace{M+M}_{2M}$$

Now $$|I_2| \le \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) |F_N(t)| \underbrace{|f(x-t)-f(x)|}_{\le 2M}\, dt$$

$$\le \underbrace{\left(\max_{\substack{|t|>\delta \\ t\in[-\pi,\pi]}} F_N(t)\right)}_{\substack{\text{can make small as desired} \\ \text{by taking } N \text{ large enough.}}} \cdot 2M \Big(\underbrace{(\pi-\delta)+(\pi-\delta)}_{2\pi-2\delta < 2\pi}\Big)$$

Make it $< \dfrac{\varepsilon/2}{(2M)(2\pi)}$. $N > N_0$

Then $|\sigma_N(x) - f(x)| \le |I_1| + |I_2| \le \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{\varepsilon}$

when $N > N_0$. ✓

Fejér's theorem #2 Trigonometric poly's are dense among 2π-periodic continuous fcns on $\mathbb{R}$.

Pf: $S_N(t)$ are trig polys. Hence so are $\sigma_N(t)$'s.

Huge consequence Bessel's ineq $\to$ $\underbrace{\text{equality as } N\to\infty}_{\text{Parseval's identity}}$.

Why $0 \le \underbrace{\Big\| f - \sum_{-N}^{N} \overset{\hat{f}(n)}{c_n} e^{inx} \Big\|^2}_{\text{best } L^2\text{-approx of form } \sum_{-N}^{N} A_n e^{inx}} \le \underbrace{\Big\| f - \underbrace{\sum_{n=-N}^{N} A_n e^{inx}}_{\text{Use } \sigma_N(x) \text{ here!}} \Big\|^2}_{\sigma_N(x)\to f \text{ unif. So this} \to 0 \text{ as } N\to\infty.}$

"See $\sum_{-N}^{N} c_n e^{inx} \longrightarrow f$ in $L^2[-\pi,\pi]$"

$$0 \le \underbrace{\Big\| f - \sum_{-N}^{N} c_n e^{inx} \Big\|^2}_{= \|f\|^2 - \sum_{-N}^{N} \frac{|c_n|^2}{2\pi}} < \varepsilon, \quad N > N_0.$$

See $\underbrace{\lim_{N\to\infty} \sum_{-N}^{N} |c_n|^2}_{\sum_{-\infty}^{\infty} |c_n|^2} = 2\pi \|f\|^2 = 2\pi \int_{-\pi}^{\pi} f(x)^2\,dx$

Parseval's ✓