Lecture 11 Solution to the Dirichlet problem on the unit disc

Parseval's: f continuous on $[-\pi, \pi]$, 2π-periodic

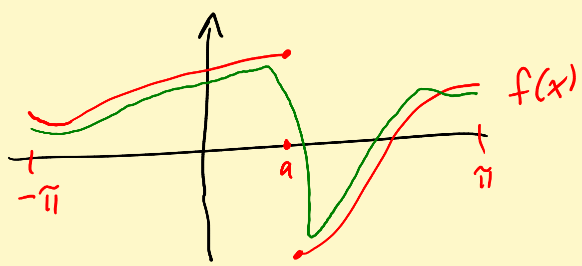
$$0 \le \underbrace{\left\| f - \sum_{-N}^{N} c_n e^{inx} \right\|^2}_{= \|f\|^2 - \frac{1}{2\pi} \sum_{-N}^{N} |c_N|^2} \le \underbrace{\left\| f - \sum_{-N}^{N} A_n e^{inx} \right\|^2}_{\text{can make } < \varepsilon \text{ by Fejér's}}$$

Bessel's, Fejér's $\Longrightarrow$ Parseval's: $\|f\|^2 = \frac{1}{2\pi} \sum_{-\infty}^{\infty} |c_n|^2$

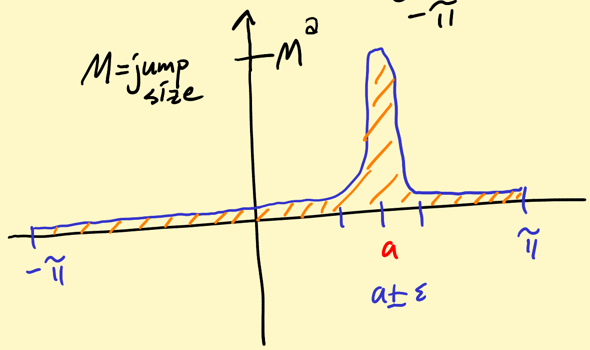
Hmmm. Bessel's holds when f is continuous, with finitely many jumps on $[-\pi, \pi]$. Does Parseval's hold?

Yes, because ...

Trick



$$\| \text{Red} - \text{Green} \|^2 = \int_{-\pi}^{\pi} (\text{Red} - \text{Green})^2 \, dx$$



Area can be made as small as desired.

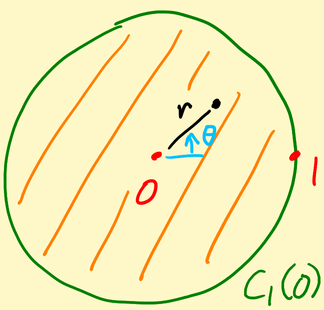
Conclusion: Parseval's holds for piecewise continuous fcns.
In particular, endpoints at $\pm\pi$ don't have to connect.

Big fact: Fourier series for f converges to f in L^2!

Riemann integration has limitations!

$\exists$ Riemann integrable fcns f_n that are a Cauchy seq in $L^2[-\pi,\pi]$ that don't converge to a Riemann integrable fcn! Need Lebesgue integral.

Dirichlet problem



$D_1(0) = \{(x,y) : \sqrt{x^2+y^2} < 1\}$

Know temp on unit circle $C_1(0)$, which is held steady. Let temp reach steady state. Heat eqn: $\underbrace{\frac{\partial u}{\partial t}}_{=0} = c^2 \Delta u$

at steady state.

Solutions to steady state heat problem are harmonic fcns, meaning that $\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\Delta} u \equiv 0$ $\quad$ $\Delta u \equiv 0$ Laplace eqn

Dirichlet problem:
1) Need $\Delta u \equiv 0$ in $D_1(0)$
2) u continuous up to $C_1(0)$
3) $u(1,\theta) = f(\theta) \leftarrow$ given fcn

$0 \le \theta \le 2\pi$

[Note: $f(0) = f(2\pi)$ so f cont on $C_1(0)$.]

(For f to be C^1-smooth on $C_1(0)$, need $f'(0) = f'(2\pi)$ too.)

MA 261: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \underbrace{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\text{Laplacian in polar coords}}$

Try "separation of variables".

Look for $u(r,\theta) = R(r)\Theta(\theta)$

want

$$\Delta u \equiv 0$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' \equiv 0$$

Divide by $R\Theta$. Multiply by r^2.

$$\boxed{\frac{r^2R'' + rR'}{R} = \frac{-\Theta''}{\Theta}} = \lambda, \text{ a const}$$

(Let $\theta = \frac{\pi}{2}$, etc.)

Two ODE's:

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ r^2R'' + rR' - \lambda R = 0 \end{cases}$$

$\leftarrow$ Euler equ

Θ problem $\quad \Theta'' + \lambda\Theta = 0$

"Periodic boundary conditions" $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$

No jumps, no kinks at $0, 2\pi$ connection.

$\underline{\lambda = 0}$ $\quad \Theta'' \equiv 0$

$\Theta(\theta) = A\theta + B \qquad \Theta(0) = B$

$\Theta(2\pi) = A \cdot 2\pi + B$

$\Theta(\theta) \equiv B$ $\quad$ see need $A=0$.

$\Theta'(0) = 0 = \Theta'(2\pi)$. Oh good!

Get possible sol^n $\boxed{\Theta(\theta) = 1}$ when $\lambda = 0$.

Write $\Theta_0(\theta) = 1$. (Take $B=1$)

$\underline{\lambda < 0}$ $\quad$ Get $\Theta(\theta) = c_1 e^{k\theta} + c_2 e^{-k\theta}$ $\quad (\lambda = -k^2)$

Ouch! Can't satisfy periodic B.C.

Get no sol^n's except the $\equiv 0$ sol^n's.

$\underline{\lambda > 0}$ $\quad$ Get periodic sol^n's exactly when $\lambda = n^2$

$$\boxed{\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta}$$

Need to solve Euler eqn for $\lambda = 0, 1^2, 2^2, 3^2, \ldots$

$\lambda = 0$ will be easy.

$\lambda \neq 0$ $\quad r^2 R'' + r R' - \lambda R = 0$

Euler's idea. Try $R(r) = r^p$ and <u>make it</u> work.