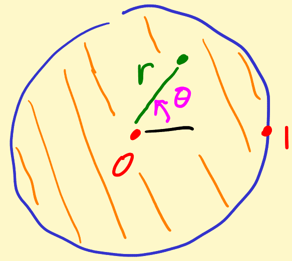
Lecture 12 Solution to the Dirichlet Problem and the Poisson kernel

Dirichlet problem on unit disc



$u(r,\theta)$ = steady state temp

$\Delta u \equiv 0$ inside

u continuous up to boundary

with boundary values prescribed : $u(1,\theta) = f(\theta)$ ← given

Θ problem: $\Theta'' + \lambda\Theta = 0$ periodic bndry cond.

Requires $\lambda = 0, 1^2, 2^2, 3^2, \ldots$

$$\Theta_0(\theta) = 1$$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

R(r) problem: Euler eqn

$$r^2 R'' + rR' - n^2 R = 0 \qquad \lambda = n^2 \quad n = 0, 1, 2, \ldots$$

Try $R(r) = r^p$ $\qquad R'(r) = p\,r^{p-1}$

$\qquad\qquad\qquad\qquad\qquad R''(r) = p(p-1)\,r^{p-2}$

Plug in. Force it!

$$r^2\left(p(p-1)r^{p-2}\right) + r\left(p r^{p-1}\right) - n^2 r^p = 0 \quad \text{← want}$$

$$\underbrace{\left[p(p-1) + p - n^2\right]}_{=0} r^p = 0$$

need $=0$

Need $\boxed{p(p-1)+p-n^2=0}$ ← characteristic eqn for Euler eqn.

$$p^2-n^2=0$$

$$\boxed{p=\pm n}$$

Get two sol^{n}s r^n, r^{-n} ← Clearly linearly independent: One is not a const multiple of the other. [or check Wronskian $\neq 0$]

$$W=\det\begin{bmatrix} r^n & r^{-n} \\ nr^{n-1} & -nr^{-n-1}\end{bmatrix}\neq 0$$

So $R(r)=Ar^n+\underbrace{Br^{-n}}$

↑ Ouch! Temp blows up at origin. Conclude that we need $B=0$ via physical considerations.

$R_n(r)=Ar^n \qquad \Theta_n(\theta)=a_n\cos n\theta+b_n\sin n\theta$

$$\boxed{u_n(r,\theta)=r^n\left(a_n\cos n\theta+b_n\sin n\theta\right)}$$

n=0: $p^2=0$. Only get $p=0$. $r^0\equiv 1$.

ODE fact: Reduction of order: Get second

$\text{sol}^n : \ln r$ ← bad at $r=0$. Toss it!

Get $R_0(r) = 1$. $\Theta_0(\theta) = 1$

So $u_0(r,\theta) \equiv 1$.

$\Delta u \equiv 0$ is a linear PDE. So Supperposition principle holds: Linear combos of sol^n's are also sol^n's.

So
$$a_0 + \sum_{n=1}^{N} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Finally, think about bndry cond:

$$u(1,\theta) = a_0 + \sum_{n=1}^{N} (a_n \cos n\theta + b_n \sin n\theta) \overset{\text{want}}{=} f(\theta)$$

Need $N \to \infty$. Then a_n, b_n's are Fourier coeff.

Solution to D. prob is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

where coeff are the Fourier coeff for $f(\theta)$.

Fascinating fact: a_0 = value of u at origin

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$ ← ave of f on bndry!

Harmonic fcns satisfy this "averaging property" on circles.

<u>Fantastic calculation</u> Turn real Fourier coeff into complex version:

$$u_N(r,\theta) = a_0 + \sum_{n=1}^{N} \underbrace{r^n\left(a_n \frac{e^{in\theta}+e^{-in\theta}}{2} + b_n \frac{e^{in\theta}-e^{-in\theta}}{2i}\right)}_{r^n\left[\underbrace{\frac{a_n - i b_n}{2}}_{c_n} e^{in\theta} + \underbrace{\frac{a_n + i b_n}{2}}_{c_{-n}} e^{-in\theta}\right]}$$

$$= a_0 + \sum_{n=-N}^{-1} r^{|n|} c_n e^{in\theta} + \sum_{1}^{N} r^n c_n e^{in\theta}$$

$$= \sum_{-N}^{N} c_n r^{|n|} e^{in\theta}$$

$$= \sum_{-N}^{N} r^{|n|} \Big(\underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-int}\, dt}_{c_n}\Big) e^{in\theta}$$

$$= \int_0^{2\pi} \underbrace{\left(\frac{1}{2\pi}\sum_{-N}^{N} r^{|n|} e^{in(\theta-t)}\right)}_{\text{Aha! Sum of } \underline{\text{two}} \text{ geometric series}} f(t)\, dt$$

$$\frac{1}{2\pi}\left(\sum_{0}^{N} + \sum_{-N}^{-1}\right)$$

$$w: \quad r^n e^{in(\theta-t)} = \Big[\underbrace{re^{i(\theta-t)}}_{w}\Big]^n$$

$$\tilde{w}: \quad r^{|n|} e^{-in(\theta-t)} = \Big[\underbrace{re^{-i(\theta-t)}}_{\tilde{w}}\Big]^n$$

Easy to sum explicitly. Do some clever algebra like we did with $D_N(t)$. Let $N \to \infty$.

Get <u>Poisson formula</u>

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|re^{i\theta} - e^{it}|^2} f(t)\, dt$$

Mind blowing fact. Forget about questionable things we did like letting $N \to \infty$, etc.

This concrete thing can be shown to solve D. problem!

Easy thing: Can take Δ under $\int$. u is harmonic.

$u(r,\theta) \to f(\theta)$: Key: Poisson kernel is a "good kernel"