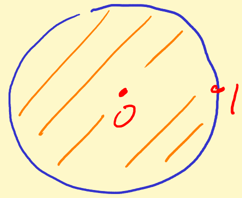
Lecture 13 The Poisson kernel

u continuous on $\overline{D_1(0)}$

$\Delta u \equiv 0$ in $D_1(0)$

$u(1,\theta) = f(\theta)$ ← given



$D_1(0)$

Dirichlet problem

Last time; Solution: $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right)$

a's, b's are the Fourier coeff for $f(\theta)$ on $[-\pi, \pi]$.

Complex form: $= \lim_{N\to\infty} \left(c_0 + \sum_{n=-N}^{-1} c_n r^{|n|} e^{in\theta} + \sum_{n=1}^{N} c_n r^n e^{in\theta} \right)$

$$= \lim_{N\to\infty} \int_0^{2\pi} \underbrace{\left(\frac{1}{2\pi} \sum_{-N}^{N} r^{|n|} e^{in(\theta - t)} \right)}_{\text{converges very quickly when } r<1.} f(t)\, dt$$

$\frac{1}{2\pi} \left(\sum_0^N + \sum_{-N}^{-1} \right)$ $\begin{cases} w = e^{i(\theta - t)} \\ \bar{w} = e^{-i(\theta-t)} \end{cases}$

$$\sum_0^N = 1 + (rw) + \cdots + (rw)^N = \frac{1-(rw)^{N+1}}{1-rw} \to \boxed{\frac{1}{1-rw}} \quad N \to \infty$$

$$\sum_{-1}^{-N} = (r\bar{w}) + (r\bar{w})^2 + \cdots + (r\bar{w})^N = (r\bar{w})\left[1 + (r\bar{w}) + \cdots + (r\bar{w})^{N-1}\right]$$

$$= (r\bar{w}) \frac{1-(r\bar{w})^N}{1-r\bar{w}} \to \boxed{\frac{r\bar{w}}{1-r\bar{w}}}$$

$N \to \infty$

So $$\sum_{-\infty}^{\infty} = \frac{1}{1-rw} + \frac{r\bar{w}}{1-r\bar{w}}$$

$$= \frac{1 - r\bar{w} + r\bar{w} - r^2\, w\cdot\bar{w}}{(1-rw)\underbrace{(1-r\bar{w})}_{\text{conjugate of } 1-rw}}$$

$w\cdot\bar{w} = |w|^2 = 1$

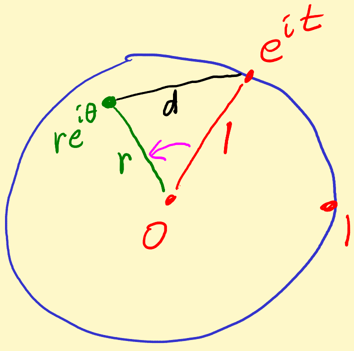
$w = e^{i(\theta-t)}$

$$= \frac{1-r^2}{|1-rw|^2} = \frac{1-r^2}{|1-re^{i(\theta-t)}|^2} \cdot \underbrace{\frac{1}{|e^{it}|^2}}_{=1}$$

$$= \frac{1-r^2}{|e^{it}-re^{i\theta}|^2}$$

Conclude that $u(r,\theta) = \int_0^{2\pi} P(r,\theta,t)\, f(t)\, dt$ where

$$P(r,\theta,t) = \frac{1}{2\pi}\,\frac{1-r^2}{|e^{it}-re^{i\theta}|^2}$$ is the <u>Poisson kernel</u>



Angle $= \theta - t$

Law of cosines

$$d^2 = 1^2 + r^2 - 2r\cdot 1\cdot \cos(\theta - t)$$

$$P(r,\theta,t) = \frac{1}{2\pi}\,\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$$

famous formula

Good kernel properties

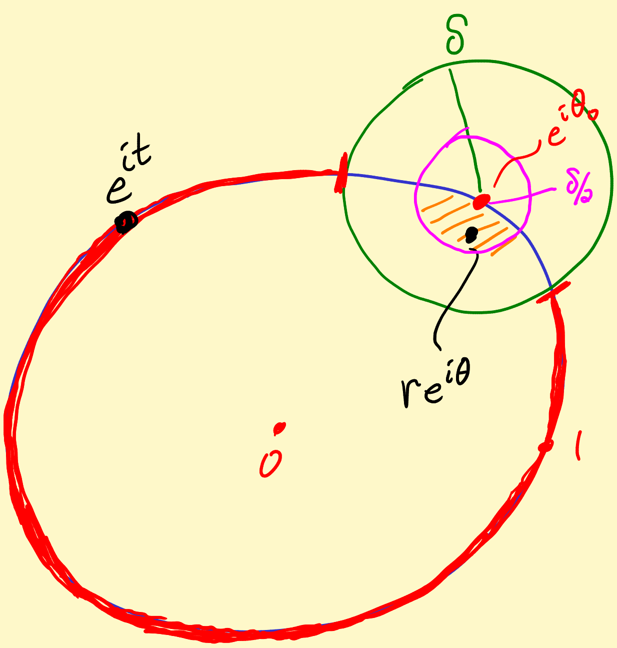
1) $P(r,\theta,t) > 0$, $0 \le r < 1$

✓ from formula

2) $\displaystyle\int_0^{2\pi} P(r,\theta,t)\,dt = 1$, $0 \le r < 1$, any θ.

Why $$\int_0^{2\pi} = \lim_{N\to\infty}\left[\underbrace{\int_0^{2\pi}\frac{1}{2\pi}\cdot 1\,dt}_{=1} + \sum_{\substack{n\neq 0\\ |n|\le N}}\underbrace{\frac{1}{2\pi}\int_0^{2\pi} r^{|n|}e^{int}\,dt}_{\text{all } =0}\right]$$

3)



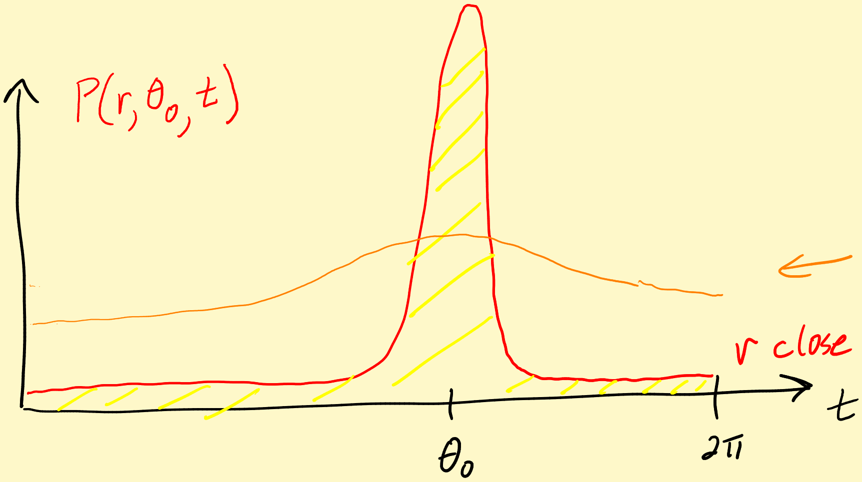
$I_\delta = \{t : e^{it} \notin D_\delta(e^{i\theta_0})\}$

$P(r,\theta,t) \to 0$ <u>uniformly</u>
when $t \in I_\delta$, $re^{i\theta} \in D_{\delta/2}(e^{i\theta_0})$
as $r \nearrow 1$

Pf: $$P(r,\theta,t) = \frac{1}{2\pi}\,\frac{1-r^2}{|re^{i\theta}-e^{it}|^2} = \frac{1}{2\pi}\,\frac{(1-r)\overbrace{(1+r)}^{<2}}{\underbrace{|re^{i\theta}-e^{it}|}_{|\cdots| \ge \delta/2}{}^2}$$

$$< \frac{1}{2\pi}(1-r)\frac{2}{(\delta/2)^2} = \frac{4}{\pi\delta^2}(1-r)$$

$\to 0$ unif as $r \nearrow 1$.



Area under both graphs is 1

4) $\Delta_{(r,\theta)} P(r,\theta,t) \equiv 0 \qquad 0 \le r < 1, \quad \text{any } t.$

Can $\Delta_{(r,\theta)}$ under $\int_0^{2\pi}$ to see $u(r,\theta)$ is harmonic in $D_1(0)$.

Now it's easy to show $u(r,\theta) \to f(\theta_0)$ as $re^{i\theta} \to e^{i\theta_0}$.

Pf

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta,t)\, f(t)\, dt$$

$$\left| u(r,\theta) - f(\theta_0) \right| = \left| u(r,\theta) - \underbrace{\left(\int_0^{2\pi} P(r,\theta,t)\, dt \right)}_{1} f(\theta_0) \right|$$

$$= \left| \int_0^{2\pi} P(r,\theta,t) \left[f(t) - f(\theta_0) \right] dt \right|$$

$$= \left| \underset{\substack{\uparrow \\ P \text{ small here}}}{\int_{I_\delta}} + \underset{\substack{\uparrow \\ [f(t)-f(\theta_0)] \text{ small here}}}{\int_{\theta_0-\delta}^{\theta_0+\delta}} \right|$$

Let $\varepsilon > 0$. f continuous. So $\exists\, \delta > 0$ such that

$$|f(t) - f(\theta_0)| < \frac{\varepsilon}{2} \quad \text{when} \quad |t - \theta_0| < \delta.$$

f continuous on $[-\pi, \pi]$, so bounded there. Say $|f| < M$.

So $\quad |f(t) - f(\theta_0)| < |f(t)| + |f(\theta_0)| < M + M = 2M$

$$\leq \left| \int_{I_\delta} + \int_{\theta_0-\delta}^{\theta_0+\delta} \right|$$

$$\leq \underbrace{\int_{I_\delta} \underset{\substack{\uparrow \\ P>0,\checkmark}}{P(r,\theta,t)} \underbrace{|f(t) - f(\theta_0)|}_{<2M} dt}_{I_1} + \underbrace{\int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta,t) \underbrace{|f(t)-f(\theta_0)|}_{<\frac{\varepsilon}{2}} dt}_{I_2}$$

$$I_1 \leq \max_{t \in I_\delta} P(r,\theta,t) \cdot 2M \cdot \underbrace{\text{Length}(I_\delta)}_{<2\pi}$$

Aha! Max $\to 0$ unif as $r \nearrow 1$. Can make $< \frac{\varepsilon}{2}$.

$$I_2 \leq \int_{\theta_0-\delta}^{\theta_0+\delta} \underbrace{P(r,\theta,t)} \frac{\varepsilon}{2}\, dt$$

P is scary big for θ close to θ_0!

$$\leq \int_0^{2\pi} P(r,\theta,t) \frac{\varepsilon}{2}\, dt = \frac{\varepsilon}{2} \cdot 1 \checkmark$$

brilliant!

Property 2!

Next time: Odds and ends, including a fun and easy demo that Fejér's $\Longrightarrow$ Weierstraß thm that polynomials are "dense" among continuous fcns on a closed interval.