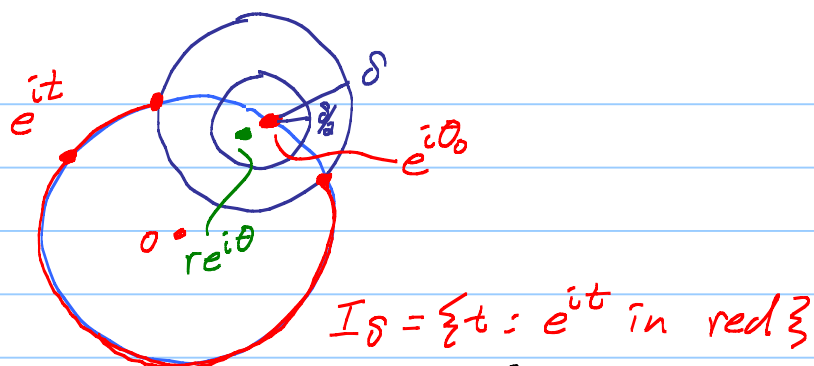


Lesson 14 Harmonic Functions



$$1) \quad \left| u(r, \theta) - f(\theta_0) \right| = \left| \int_0^{2\pi} P(r, \theta, t) [f(t) - f(\theta_0)] dt \right|$$

$$= \left| \int_{I_\delta} \underbrace{P(r, \theta, t)}_{\text{small!}} [f(t) - f(\theta_0)] dt + \int_{\theta_0 - \delta}^{\theta_0 + \delta} \underbrace{P}_{\text{small!!}} [f - f(\theta_0)] dt \right|$$

Conclude: $u(r, \theta)$ is continuous up to the boundary.

To make it continuous on $\overline{D_1(0)}$, define $u(1, \theta) = f(\theta)$.

Finally: $\Delta_{(r, \theta)} u(r, \theta) = \Delta \int = \int_0^{2\pi} \underbrace{\Delta_{(r, \theta)} P(r, \theta, t)}_{\equiv 0 \quad r < 1} f(t) dt$

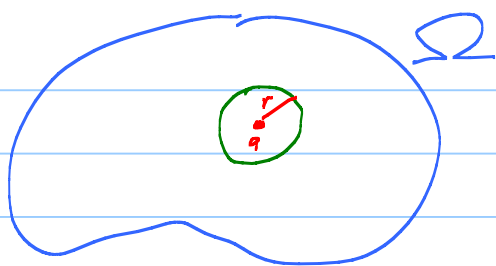
So u is harmonic on $D_1(0)$.

This proof is "Schwarz's theorem"

Thing to notice $P(0, \theta, t) = \frac{1}{2\pi} \frac{1 - 0^2}{|0 - e^{it}|} = \frac{1}{2\pi}$

$\underbrace{u(0)}_{\text{temp at } 0} = \int_0^{2\pi} \frac{1}{2\pi} \cdot f(\theta) d\theta = \text{ave of } f \text{ on unit circle}$

Fact: Harmonic fns satisfy the "averaging property"



$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta$$

Why: Green's identities can be used to show this (later).

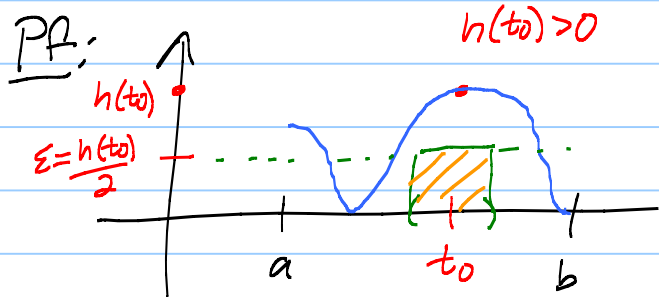
See link to "Something about Poisson and Dirichlet" on MA 428 home page.

Fact: Ave property $\Rightarrow$ Maximum principle

Calculus lemma: $0 = \int_0^{2\pi} \sin t \, dt$ does not imply that

$\sin t \equiv 0$, but if $h(t)$ is continuous and $h(t) \geq 0$

and $\int_a^b h(t) \, dt = 0$, then $h(t) \equiv 0$.



$\exists \delta > 0$ such that

$$|h(t) - h(t_0)| < \epsilon$$

if $|t - t_0| < \delta$.

Aha! $\int_a^b h(t) \, dt > \text{area}$ 

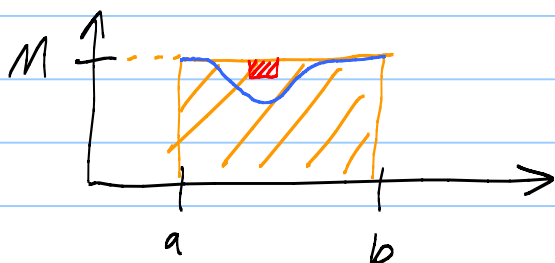
$\swarrow \searrow$ ϵ -contradiction

No such t_0 . $h(t) \equiv 0$.

Lemma: If h is cont on $[a, b]$ and $h(t) \leq M$ there, then

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if $\int_a^b h(t) dt = M(b-a)$, then $h(t) \equiv M$.

Pf: Apply first lemma to $\tilde{h}(t) = M - h(t)$.



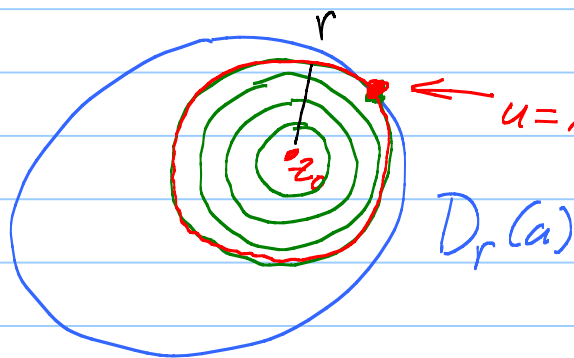
Max principle for harmonic fcn's: If u is cont on $\overline{D_r(a)}$ and harmonic inside, then u assumes its max M at a pt on the boundary.

Pf: $\overline{D_r(a)}$ is a closed and bounded set (compact).

Know a cont fcn on a closed bdd set assumes its max at a pt, $z_0 \in \overline{D_r(a)}$.

Case: z_0 in bndry. Done!

Case: $z_0 \in D_r(a)$ (inside).



$u(z_0) = M$, max
 $\leftarrow u = M$ here! on bndry.

Taking limits of expanding circles $\Rightarrow$

$$\underbrace{u(z_0)}_M = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{u(z_0 + re^{i\theta})}_{\leq M} d\theta \leq \underbrace{\frac{1}{2\pi} \cdot M \cdot 2\pi}_M$$

Calc lemma $\Rightarrow u(z_0 + re^{i\theta}) \equiv M$ on red circle.

Big application: Solⁿ to D. problem is unique!

Pf: Two solⁿs u_1 and u_2 . Then

$u_1 - u_2$ is a harmonic fcn, cont up to boundary and $\equiv 0$ on bndry.

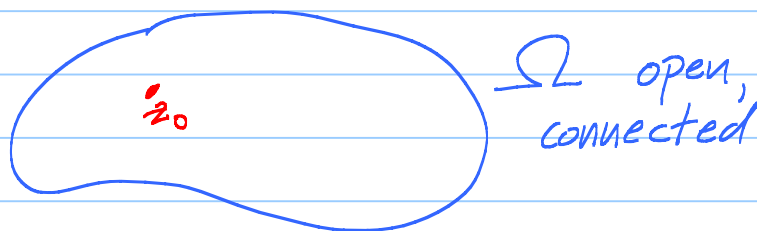
$$\text{Max princ} \Rightarrow \frac{\text{Max}_{\overline{D(0)}} (u_1 - u_2)}{} = \text{Max}_{C_1(0)} (u_1 - u_2) = 0$$

$$\text{So } u_1 - u_2 \leq 0.$$

Repeat for $u_2 - u_1$. See $u_2 - u_1 \leq 0$ too.

Conclude $u_1 \equiv u_2$.

Second version of max princ:



$z_0 = \text{"hot spot"}$

If a harmonic fcn has a local max at a pt $z_0 \in \Omega$, then it must be constant on Ω .

Local max: $\exists D_\varepsilon(z_0) \subset \Omega$ such that $u(z) \leq u(z_0)$ on $D_\varepsilon(z_0)$.

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Another biggy:

Know $u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} f(t) dt$

Wow! Can differentiate under $\int$ infinitely many times in r and θ .

Consequently, harmonic fns are C^∞ -smooth!

Defⁿ: u is harmonic means that u is twice continuously diffⁿ ble and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$.

Big fact: If a continuous fcn satisfies the ave prop on an open set, it is harmonic there.