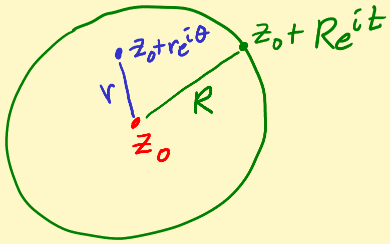
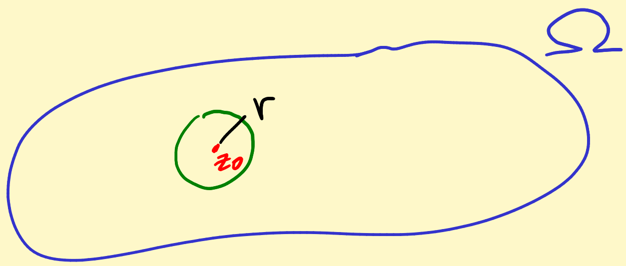
Lecture 15 Harmonic functions and the averaging property

Poisson formula on a disc $D_R(z_0)$



$$u(z_0+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-t)}\, u(z_0+Re^{it})\,dt$$

Theorem A continuous function that satisfies the averaging property is harmonic!



$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt$$

Pf: Key idea: We've seen that ave prop $\Rightarrow$ Max princ.

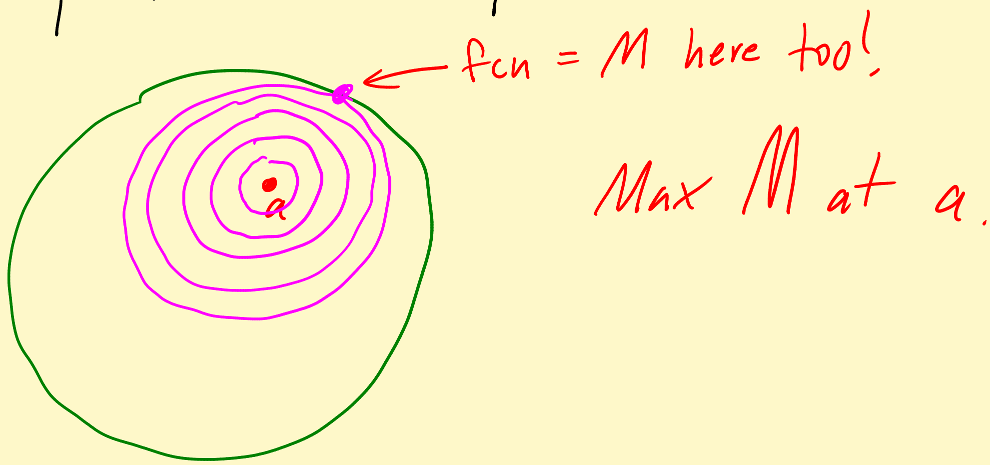
Clever idea: Restrict attention to a disc $\overline{D_R(z_0)} \subset \Omega$.

Define $U(z)$ = Poisson integral of boundary values of u on $C_R(z_0)$.

Know U is continuous on $\overline{D_R(z_0)}$, has boundary values $U(z_0+Re^{it}) = u(z_0+Re^{it})$, and U is harmonic inside $D_R(z_0)$ $\leftarrow$ and so it satisfies the ave property inside $D_R(z_0)$ too!

Aha! $u-U$ satisfies ave prop and is continuous up to the boundary and is <u>zero</u> there.

Picture for proof of Max princ:



For $u - U$, that max on bndry is 0.

Conclude $u - U \le 0$ on inside too.

Repeat argument using $U - u$ in place of $u - U$.

see $U - u \le 0$ too.

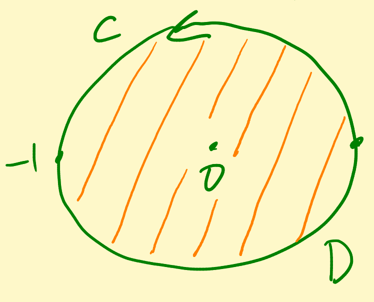
So $u \equiv U$ in $D_R(z_0)$. Aha! u is harmonic

(and consequently C^∞-smooth) on $D_R(z_0)$.

Since $D_R(z_0)$ was arbitrary, see u is harmonic on Ω.

<u>Importance of Green's theorem and Green's identities</u>

Green's thm is <u>easy</u> and <u>elementary</u> on a disc. (Just the fund thm calc.)

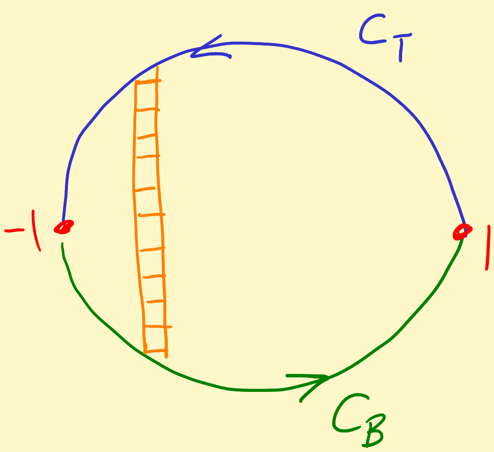


$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Pf Parametrize C by $(x(t), y(t))$ as $a \le t \le b$

EX: $(\cos t, \sin t)$, $0 \le t \le 2\pi$

$$\int_C P\,dx + Q\,dy = \int_a^b P(x(t), y(t))\, \underbrace{\dot{x}(t)\,dt}_{\underbrace{\frac{dx}{dt}dt}_{dx}} + \int_a^b Q(x(t), y(t))\, \underbrace{\dot{y}(t)\,dt}_{dy}$$



$-C_T$: $\begin{cases} x(t) = t \\ y(t) = \sqrt{1-t^2} \end{cases}$ $-1 \le t \le 1$

C_B : $\begin{cases} x(t) = t \\ y(t) = -\sqrt{1-t^2} \end{cases}$ $-1 \le t \le 1$

Claim: $\underbrace{\int_C P\,dx}$ $= -\iint_D \frac{\partial P}{\partial y}\,dx\,dy$

$$\int_C = \left(\int_{C_B} + \int_{C_T}\right) = \left(\int_{C_B} - \int_{-C_T}\right)$$

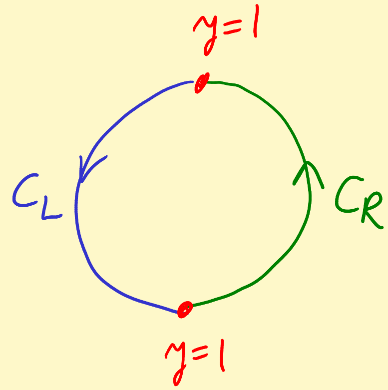
$$= \int_{-1}^{1} P(t, h_B(t)) \cdot \underset{\substack{\uparrow \\ \dot{x}(t)=1 \\ x(t)=t}}{1} \cdot dt - \int_{-1}^{1} P(t, h_T(t)) \cdot 1 \cdot dt$$

Hmmm $= -\int_{-1}^{1} \left[P(t, h_T(t)) - P(t, h_B(t)) \right] dt$ ← change t to x

$$= -\int_{-1}^{1} \underbrace{\left[P(x, h_T(x)) - P(x, h_B(x)) \right]}_{\int_{h_B(x)}^{h_T(x)} \frac{\partial P}{\partial y}(x,y)\, dy} dx$$

$$= -\iint_D \frac{\partial P}{\partial y}\, dy$$

The other half



$C_R : \begin{cases} y(t) = t, & -1 \le t \le 1 \\ x(t) = \sqrt{1-t^2} \end{cases}$

$-C_L : \begin{cases} y(t) = t \\ x(t) = -\sqrt{1-t^2} \end{cases}$

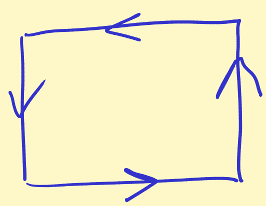
$$\int_C Q\, dy = \left(\int_{C_R} - \int_{-C_L} \right) Q\, dy$$

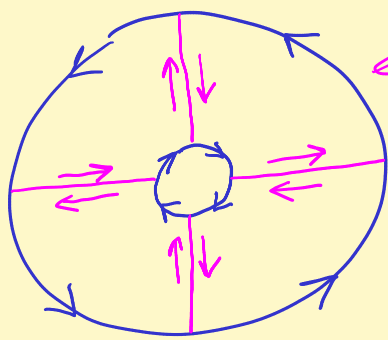
$$= \int_{-1}^{1} \left[Q(g_R(t), t) - Q(g_L(t), t) \right] dt$$ ← change t to y

$$= \int_{-1}^{1} \left(\int_{g_L(y)}^{g_R(y)} \frac{\partial Q}{\partial x}(x,y)\,dx \right) dy \quad \checkmark$$

<u>Remark</u> Can do same proof when region is bounded by a curve with a top and bottom and a left and right.

<u>EX</u>

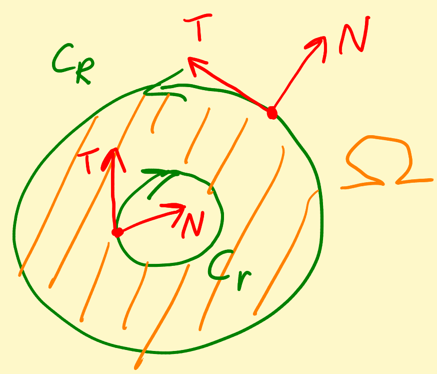




each "quad" curve has top/bottom, left/right property.

Purple cut integrals <u>cancel</u> when added up.

<u>Green's thm for an annulus</u>



$$\left(\int_{C_R} - \int_{C_r} \right) P\,dx + Q\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

"Orientation": Outward pointing normal vector is to the <u>right</u> of the <u>tangent vector</u>.

Green's thm $\Longleftrightarrow$ 2-D Divergence theorem

$$\int_C \vec{F} \cdot \hat{n} \, ds = \iint_\Omega \underbrace{\operatorname{div} \vec{F}}_{\nabla \cdot \vec{F}} \, dx\, dy$$

$$\vec{F} = F_1 \hat{\imath} + F_2 \hat{\jmath} \qquad \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$$

Spivak: Calculus on manifolds