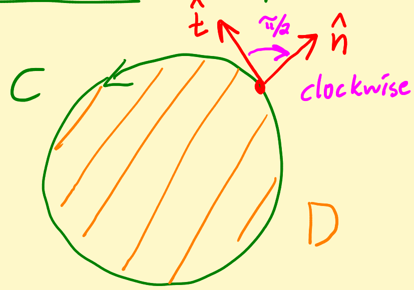
Lecture 16 Importance of Green's theorem



$$\int_C \underbrace{P\,dx + Q\,dy}_{} = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{}$$

Stokes: ω

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

$$d\omega = dP\wedge dx + dQ \wedge dy$$
$$= \left(-\frac{\partial P}{\partial y} dx\wedge dy + \frac{\partial Q}{\partial x} dx \wedge dy\right)$$

<u>Divergence theorem</u> (Gauss' thm)

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \operatorname{Div}\vec{F}\,dx\,dy$$

$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$ $\qquad$ $\operatorname{Div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$

Aha! Take $\begin{cases} F_1 = Q \\ F_2 = -P \end{cases}$

$$= \underbrace{\int_C (-F_2)dx + (F_1)dy}_{\text{This is the "flux"}}$$

$$= \int_C \vec{F}\cdot\hat{n}\,ds$$

<u>why</u> $\int_C -F_2\,dx + F_1\,dy$ is the flux.

$C: (x(t), y(t)), \quad 0 \le t \le L$

where $t =$ arc length

$\hat{t} = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}$ ← unit vector in tangent direction

Now
$$\int_C -F_2\,dx + F_1\,dy = \int_0^L \left[-F_2(x(t),y(t))\,\dot{x}(t) + F_1(x(t),y(t))\,\dot{y}(t)\right] dt$$

$$= \int_0^L (F_1, F_2)\cdot \underbrace{(\dot{y}(t), -\dot{x}(t))}_{\text{Claim: unit normal vector}}\,dt$$

<u>Check</u> $\underbrace{(\dot{x}, \dot{y})}_{\hat{t}} \cdot \underbrace{(\dot{y}, -\dot{x})}_{} = 0$

orthogonal ✓ $\qquad$ unit vector ✓ (same length as $\hat{t}$)

Claim $(\dot{y}, -\dot{x})$ points out

Nice way to see: Represent vectors in $\mathbb{R}^2$ by complex #'s.

$$\hat{t} \sim \dot{x} + i\dot{y}$$

$$\hat{n} \sim \underbrace{e^{-i\pi/2}} \hat{t}$$

$$= -i\,(\dot{x} + i\dot{y})$$

$$= \underbrace{\dot{y} - i\dot{x}}_{=\ \hat{n}\ \text{pointing out.}} \quad \text{Yes!}$$

$$e^{i\alpha}\left(re^{i\theta}\right)$$

$$re^{i(\theta+\alpha)}$$

rotates by α in counterclockwise direction

Green's 1st identity

$$\int_C u \frac{\partial v}{\partial n}\, ds = \iint_D \nabla u \cdot \nabla v + u \Delta v \, dx\, dy$$

Why: Let $\vec{F} = u \nabla v$ in divergence thm.

$$\int_C \vec{F} \cdot \hat{n}\, ds = \int_C u\underbrace{(\nabla v \cdot \hat{n})}_{\frac{\partial v}{\partial n}}\, ds = \iint_D \underbrace{\operatorname{Div} \vec{F}}_{\operatorname{Div}(u\nabla v)}\, dx\, dy$$

$$\operatorname{Div}(u\nabla v) \underset{\text{check}}{=} \nabla u \cdot \nabla v + u \Delta v \ \checkmark$$

Green's 2nd identity

$$\int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds = \iint_D (u\Delta v - v\Delta u)\, dx\, dy$$

Why

$$\int_C u \frac{\partial v}{\partial n}\, ds = \iint_D \nabla u \cdot \nabla v + u \Delta v \;\; dx\,dy$$

$$\Vert$$

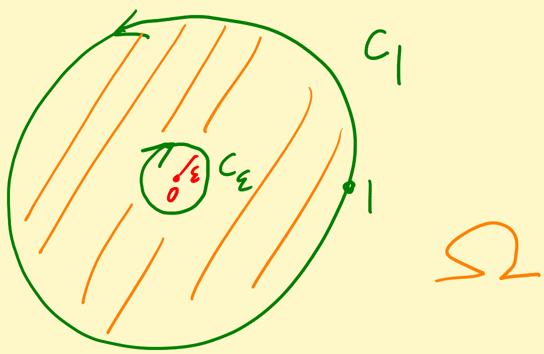
$$- \quad \int_C v \frac{\partial u}{\partial n}\, ds = \iint \nabla v \cdot \nabla u + v \Delta u \;\; dx\,dy$$

✓

Green used his 2nd identity to show that harmonic fcns satisfy the ave prop.

C^2-smooth, $\Delta \equiv 0$

Pf



$u =$ harmonic fcn

$v = \ln r \leftarrow$ harmonic

$\ln \sqrt{x^2+y^2}$

$= \frac{1}{2} \ln (x^2+y^2)$.

Check that $\Delta \equiv 0$.

$$\int_{C_1 \cup C_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \iint_\Omega u \underbrace{\Delta v}_{0} - v \underbrace{\Delta u}_{0} \;\; dx\,dy = 0$$

$$\int_{C_1} u \underbrace{\frac{\partial v}{\partial n}} - \overset{\ln 1 = 0}{v} \frac{\partial u}{\partial n} \;\; ds = \int_{-C_\varepsilon} u \underbrace{\frac{\partial v}{\partial n}}_{\frac{1}{\varepsilon}} - \overset{\ln \varepsilon}{v} \underbrace{\frac{\partial u}{\partial n}}_{\text{bounded}} \; ds$$

$v = \ln r$

$\frac{\partial v}{\partial n} = \frac{1}{r} \Big|_{r=1} = 1$

$$\int_{-C_\varepsilon} \frac{1}{\varepsilon} \cdot u \; ds - \int_{-C_\varepsilon} \ln \varepsilon \, \overbrace{\underbrace{(\text{Bounded})}_{< M}}^{\frac{\partial u}{\partial n}} \, ds$$

$$\int_{C_1} u \, ds = \underbrace{\int_0^{2\pi} u(\varepsilon e^{it}) \frac{1}{\varepsilon}(\varepsilon \, dt)}_{\to 2\pi \, u(0)} + \text{something}$$

$\to 0$ as $\varepsilon \to 0$

$\to 2\pi \; u(0)$ as $\varepsilon \to 0$ because u continuous

$$\left| -\int_{C_{-\varepsilon}} \ln\varepsilon \,(\text{Bounded})\, ds \right| \le (\ln\varepsilon)\, M \cdot \underbrace{\text{Length}(C_{-\varepsilon})}_{2\pi\varepsilon}$$

Aha! L'H shows $\varepsilon \ln\varepsilon \to 0$ as $\varepsilon \to 0$.

$$\varepsilon \ln\varepsilon = \frac{\ln\varepsilon}{\left(\frac{1}{\varepsilon}\right)} \quad \frac{-\infty}{\infty} \text{ form}$$

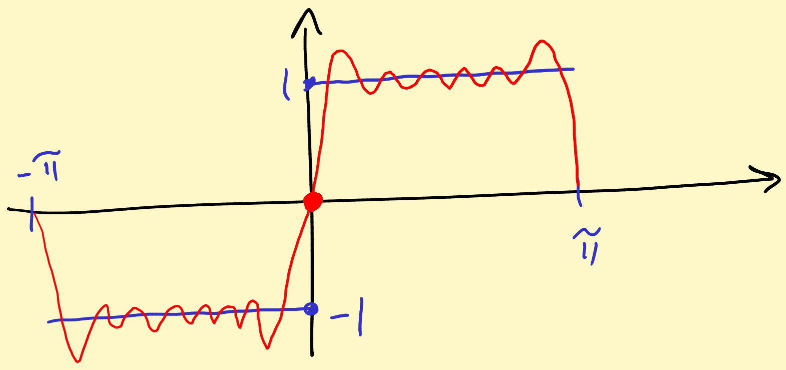
$\varepsilon \to 0$; Get $\displaystyle\int_{C_1} u \, ds = 2\pi \, u(0)$. Ave prop. ✓

Did we really need to do this? Yes. I showed the $\underbrace{\text{Poisson integral}}_{= \text{ave on } C_1 \text{ at } 0.}$ solved the Dirichlet problem.

Hang up. Need to know general harmonic fcns satisfy the ave prop to be able to show uniqueness of solution to the Dirichlet via maximum princ (which follows from the ave prop!).

Why do Fourier series converge to midpoint of jumps?

Fact Understanding a model case is key



$f(x)$ odd.
All cosine terms 0.

$a_0 = \text{ave} = 0$

$\sin nx = 0,\ x = 0.$