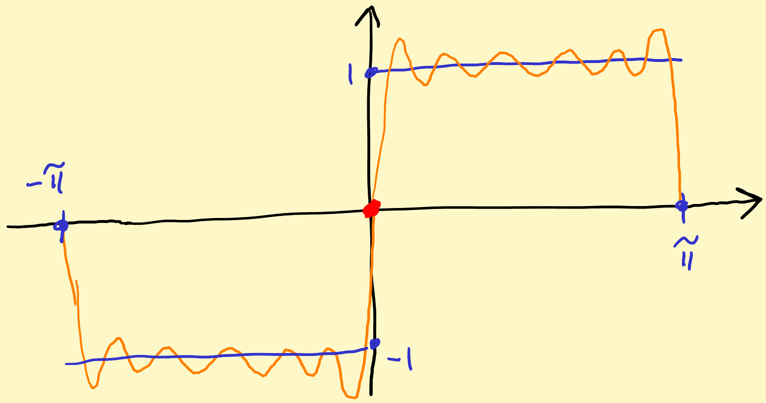
Lecture 17 The hot wire problem

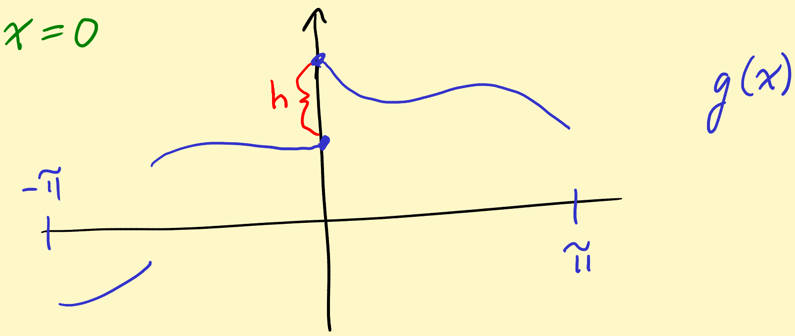
Why do Fourier series converge to the midpoint of jumps?

Test case:



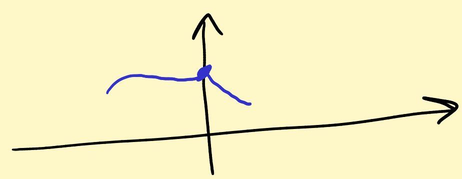
$$J(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Case Jump at $x = 0$



Aha! $g(x) - \frac{h}{2} J(x)$ has no jump at $x=0$!

But it might have a "kink" at $x = 0$



Know: At a kink of a piecewise C^1-smooth fcn, F.S. converges to the value of the function.

So $\underbrace{\text{F.S. of } g - \frac{h}{2} J}_{(\text{F.S. } g) - \frac{h}{2}(\text{F.S. } J)}$ converges to $\left(\lim_{x \to 0^-} g(x)\right) + \frac{h}{2}$ at $x = 0$

So, at $x = 0$, $\left(\lim \text{F.S.} g \text{ at } x=0\right) - \frac{h}{2} \cdot 0 = \left(\lim_{x \to 0^-} g(x)\right) + \frac{h}{2}$

Conclude that F.S. for g converges to midpoint of jump of g at $x=0$ (which is $\lim_{x\to 0^-} g(x) + \frac{h}{2}$).

How to extend idea to jumps at $x \neq 0$ pts.

Step 1 Extend f to be 2π-periodic.

Step 2 Think about "translations" of fcn

$$(T_{\underline{\underline{a}}} f)(x) = f(x - \underline{\underline{a}})$$

(Moves graph of f a distance $\underline{\underline{a}}$ to right.)

Guess: Fourier series for $(T_a f)$ is T_a of F.S. for f.

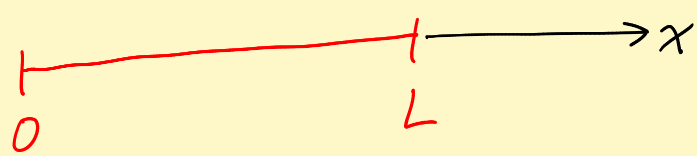
Correct! Easy to check. Need formulas for

$$\cos n(x-a), \quad \sin n(x-a)$$

and change $x - a = u$ in formulas for coeff.

Follows that same behavior at $x=0$ holds at $x=a$.

Hot wire problem



$u(x,t)$ = temp at x at time t.

Heat eqn: $\quad \dfrac{\partial u}{\partial t} = c\dfrac{\partial^2 u}{\partial x^2} \qquad$ PDE

Initial condition: temp of wire at time $t=0$ $\quad u(x,0) = f(x)$ ← known

Boundary conditions: $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ Homogeneous boundary conditions

refrigerators on both ends

Sol^n Try $u(x,t) = X(x)T(t)$

PDE $\dfrac{\partial u}{\partial t} \overset{\text{want}}{=} c\,\dfrac{\partial^2 u}{\partial x^2}$

$$X T' = c X'' T$$

$$\boxed{\frac{1}{c}\cdot\frac{T'}{T} = \frac{X''}{X} = \lambda}$$

set $x = \frac{L}{2}$. See left side const. Right side must be same const.

T-prob $\quad T' - \lambda c T = 0 \quad$ easy

$$T' = \lambda c T$$

$$\boxed{T(t) = K e^{\lambda c t}}$$

X-prob $\quad$ Boundary cond $\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(L,t) = X(L)T(t) = 0 \end{cases}$

want

Don't want $T(t) \equiv 0$. We know $u(x,t) \equiv 0$ is a sol^n.

So we must have $\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$

$$\boxed{X'' - \lambda X = 0 \quad \text{with} \begin{cases} X(0)=0 \\ X(L)=0 \end{cases}}$$

<u>Case $\lambda > 0$</u> $\quad \lambda = k^2 \quad (k>0)$

$$X'' - k^2 X = 0 \quad \left[X(x) = e^{rx}\right]$$

$$r^2 - k^2 = 0$$

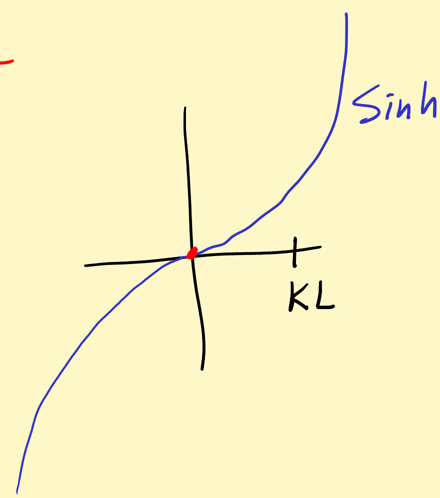
$$r = \pm k$$

$$X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

1) $X(0) = c_1 + c_2 \overset{\text{want}}{=} 0 \qquad \boxed{c_2 = -c_1}$

So $X(x) = c_1\left(e^{kx} - e^{-kx}\right)$

2) $X(L) = c_1 \underbrace{\left(e^{kL} - e^{-kL}\right)}_{= 2\sinh \underbrace{kL}_{>0}} \overset{\text{want}}{=} 0$



Get $c_1 = 0$ too. Ouch.

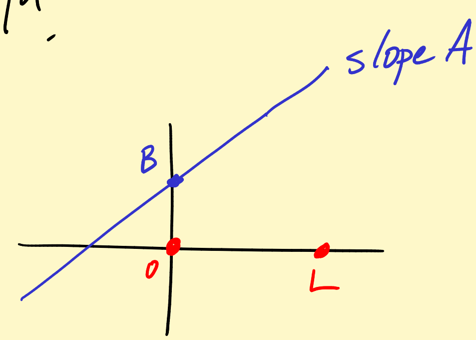
Only get $X(x) \equiv 0$ sol^n.

<u>Case $\lambda = 0$</u>

$$X'' = 0$$

$$X'(x) = A$$

$$X(x) = Ax + B$$



A, B must both be zero! $X \equiv 0$ again.

<u>Case $\lambda < 0$</u> $\quad \lambda = -k^2 \quad (k>0)$

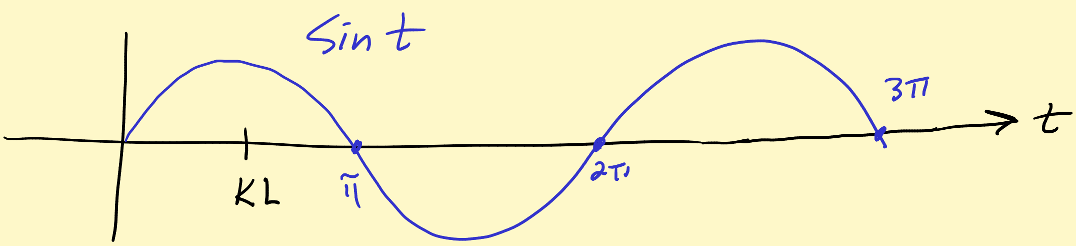
$$r^2 + k^2 = 0$$

$$X(x) = c_1 \cos kx + c_2 \sin kx$$

1) $X(0) = c_1 \overset{\text{want}}{=} 0$ $\boxed{c_1 = 0}$

So $X(x) = c_2 \sin kx$

2) $X(L) = c_2 \sin kL \overset{\text{want}}{=} 0$



Don't want $c_2 = 0$ too. Must have $\sin kL = 0$

Need $kL = n\pi$ for $n = 1, 2, 3, \ldots$

$\boxed{k = \frac{n\pi}{L}}$ $\lambda = -k^2 = -\left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$

$c_1 = 0$. Take $c_2 = 1$. (Later, we will take linear combos.)

Got $u_n(x,t) = e^{-c\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$

Key facts. PDE is linear. B.C.'s are homog.

Can add these sol^{n}s together and get more sol^{n}s to PDE + B.C.

Finally, worry about initial cond.

$$u(x,t) = \sum_{n=1}^{N} A_n e^{-c\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

Hmmm. Need $N \to \infty$ if $f(x)$ is not a trig poly with only sine terms.

Initial cond $\quad u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \overset{\text{want}}{=} f(x)$

Two things we could do: 1) Show $\sin\frac{n\pi x}{L}$ are $\perp$ on $[0,L]$. Invent a new Fourier sine series on $[0,L]$.

2) Or scale $[0,\pi]$ to $[0,L]$ and change variables in F. Sine series formulas

$$[0,\pi] \xrightarrow{x \mapsto \frac{Lx}{\pi}} [0,L]$$

Get $\quad \boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$ Done!