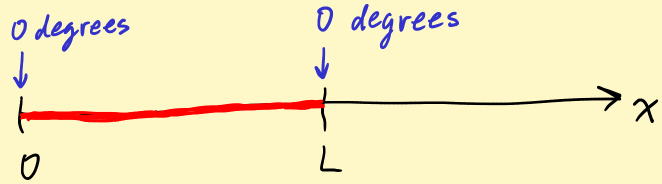
Lecture 18 The heat equation



$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \qquad \text{PDE}$$

$$\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases} \qquad \text{B.C.}$$

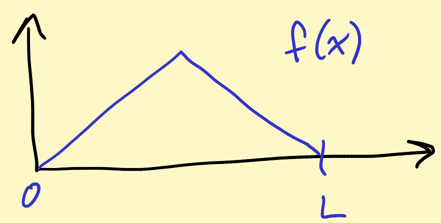
$$u(x,0) = f(x) \qquad \text{I.C.}$$

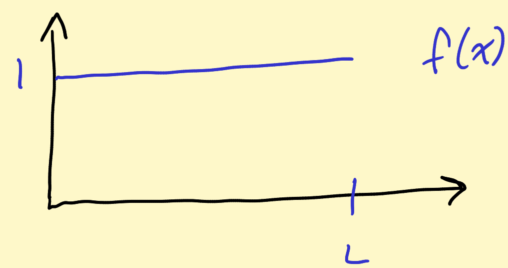
given

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-c\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}}$$

where $$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Two problems

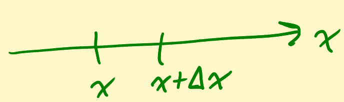




See Maple demo

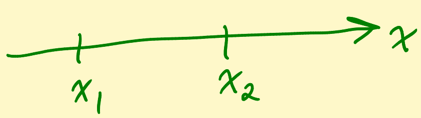
How to derive the heat eqn

(Stein does the infinitesimal version:



)

"Control volume" method.

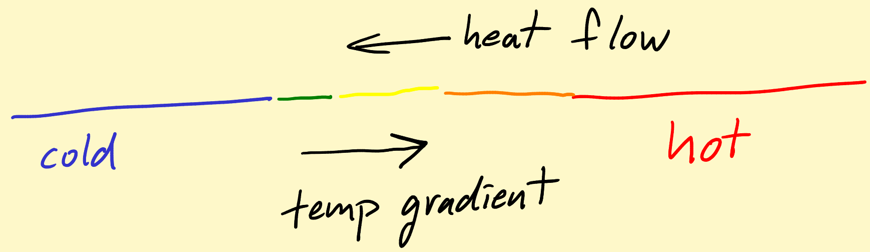


1 calorie is the heat energy required to raise one gram of water one degree Celcius (= kelvin).

$$\text{Heat in } (x_1, x_2) = K\int_{x_1}^{x_2} u(x,t)\,dx$$

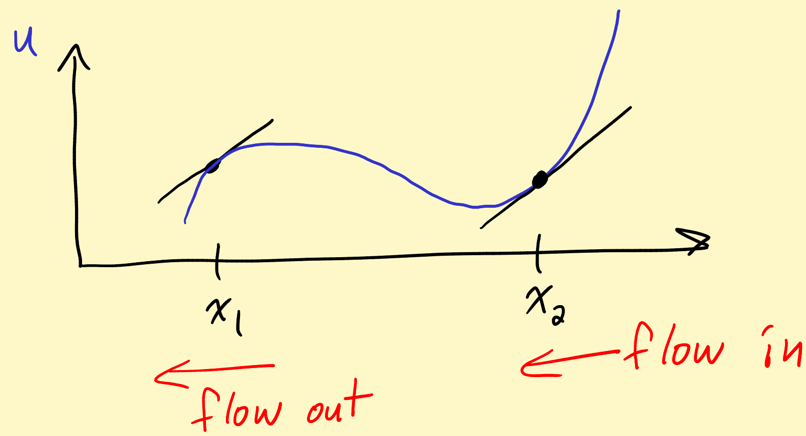
Rate it changes $= \dfrac{\partial}{\partial t}\left[k\displaystyle\int_{x_1}^{x_2} u(x,t)\,dt\right] = k\displaystyle\int_{x_1}^{x_2} \dfrac{\partial u}{\partial t}(x,t)\,dx$

Kelvin's law



$$(\text{Rate of heat flow}) = -\,c\,(\text{temp gradient})$$

EX

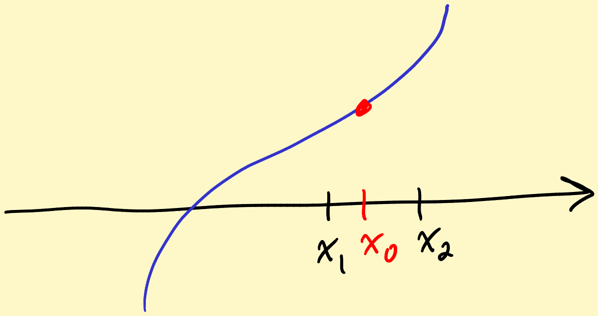


$$\left(\begin{array}{c}\text{Rate of change}\\ \text{of heat in } (x_1, x_2)\end{array}\right) = k\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx = \underbrace{c\,\frac{\partial u}{\partial x}(x_2,t) - c\,\frac{\partial u}{\partial x}(x_1,t)}_{c\int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}(x,t)\,dx}$$

Fund Thm Calc

So $$\int_{x_1}^{x_2}\left[\underbrace{k\,\frac{\partial u}{\partial t} - c\,\frac{\partial^2 u}{\partial x^2}}_{\varphi(x,t)}\right]dx = 0$$

Aha! x_1 and x_2 can be anything!

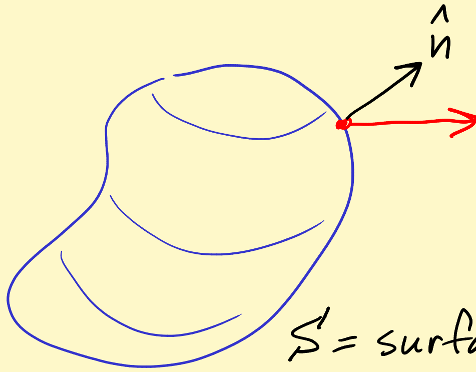


$\varphi(x_0, t_0) > 0$

Get $\int_{x_1}^{x_2} \neq 0$

Conclude $[\qquad] \equiv 0$ $\qquad \frac{\partial u}{\partial t} = \frac{c}{k} \frac{\partial^2 u}{\partial x^2}$

3-D version



∇u Temp gradient

Control volume V

S = surface of V

$$\begin{pmatrix} \text{Rate of change} \\ \text{of heat in } V \end{pmatrix} = \frac{\partial}{\partial t}\left(k \iiint_V u(x,t)\, dV_x \right)$$

$$= \text{Net heat flow into } V$$

$$= c \iint_S \nabla u \cdot \hat{n} \underbrace{d\sigma}_{\text{area}}$$

Flux of ∇u

$$\underset{\text{Div Thm}}{=} c \iiint_V \underbrace{\text{Div}(\nabla u)}_{\Delta u}\, dV$$

$$0 = \iiint_V \Big[\underbrace{k \frac{\partial u}{\partial t} - c\,\Delta u}_{\text{must } \equiv 0 \text{ because } V \text{ arbitrary.}} \Big]\, dV$$

Other heat problems

1) Homog boundary conditions: Slap on refridgerators.

2) Insulated end: No heat flowing. Kelvin $\Rightarrow$

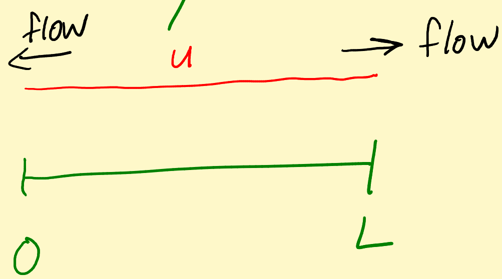
Temp gradient $=0$: $\frac{\partial u}{\partial x}(L,t)=0$ for all t.

right end insulated.

3) Radiative boundary condition

(u in Kelvin is always ≥ 0)

$\frac{\partial u}{\partial x}(0,t)=cu(0,t)$



$\frac{\partial u}{\partial x}(L,t)=-cu(L,t)$