Lecture 19 Some of the best "good" Kernels

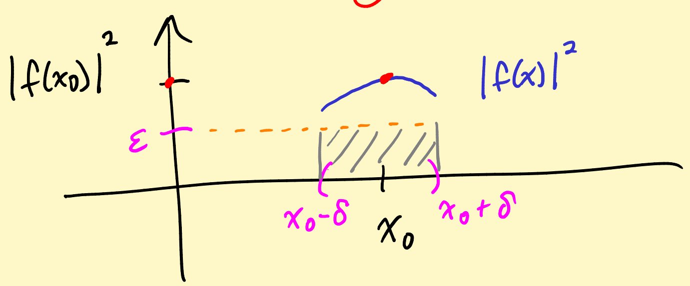
Midterm exam practice material on home page

Fact If f is piecewise continuous on $[-\pi, \pi]$ and $\hat{f}(n) = 0$ for all n, then $f(x_0) = 0$ at all points x_0 where f is continuous.

My proof: Parseval's identity:

$$\underbrace{\sum_{n=-\infty}^{\infty} |\underbrace{\hat{f}(n)}_{c_n}|^2}_{=0} = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

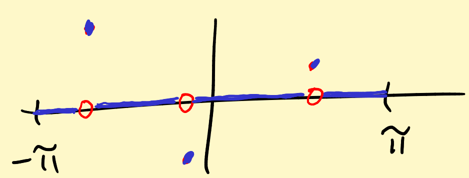
when f is piecewise continuous.



$$\varepsilon = \frac{|f(x_0)|^2}{2}$$

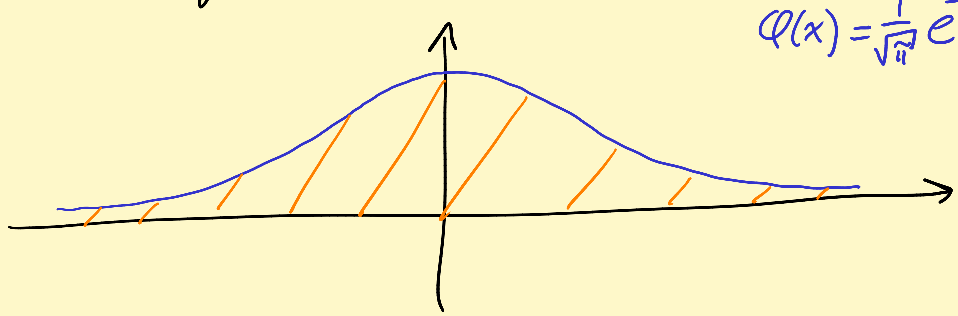
See $\displaystyle\int_{-\pi}^{\pi} |f(x)|^2\,dx >$ area rectangle $2\delta\varepsilon > 0$ ↯

Graph of f

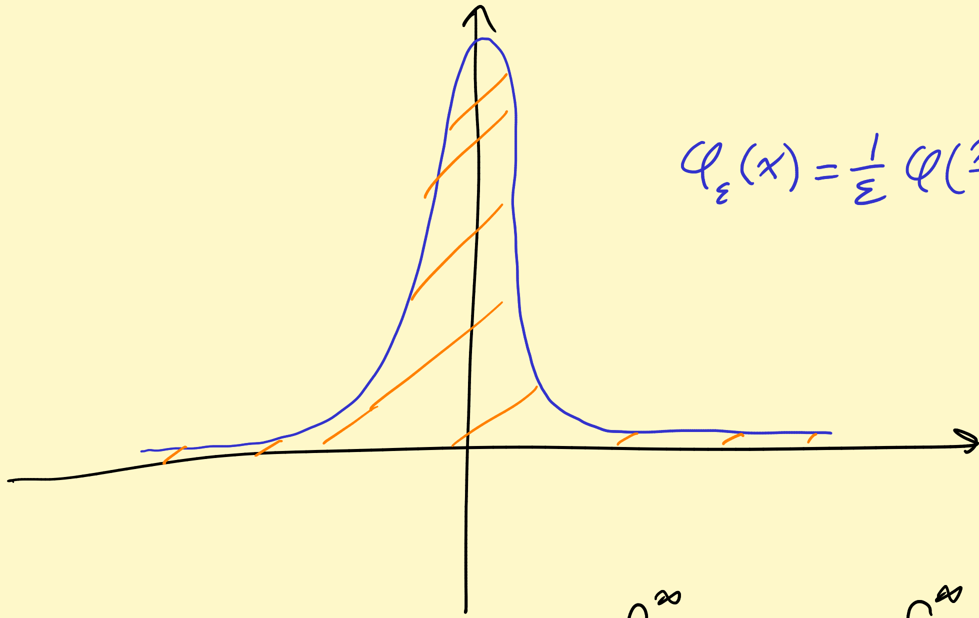


Most famous "good" kernel

$$\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$



$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$$



$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$

Change of var $u = \frac{x}{\varepsilon}$ shows $\displaystyle\int_{-\infty}^{\infty} \varphi_\varepsilon(x)\,dx = \int_{-\infty}^{\infty} \varphi(u)\,du = 1$

Famous calculation: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

1st quadrant

$$= \int_0^\infty \int_0^\infty \underbrace{e^{-x^2} e^{-y^2}}_{e^{-r^2}} \; \underbrace{dx\,dy}_{r\,dr\,d\theta}$$

$$= \int_0^{\pi/2} \int_0^\infty r e^{-r^2}\,dr\;d\theta$$

$u = r^2$

$du = 2r\,dr$

$$= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-u}\right]_{u=0}^{\infty} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} \, d\theta = \frac{\pi}{4}$$

Finally $\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \left(2\int_0^{\infty} e^{-x^2} dx \right)^2 = 4 \cdot \frac{\pi}{4} = \pi$ ✓

Famous use of φ_ε

Recall, Fejér kernel $\quad \int_{-\pi}^{\pi} \underbrace{\overset{\text{Fejér kernel}}{F_N(t)} f(x-t)}_{2\pi\text{-periodic}} dt$ convolution

On $\mathbb{R}$: $\quad (\varphi_\varepsilon * f)(x) = \int_{-\infty}^{\infty} \varphi_\varepsilon(t) f(x-t) \, dt$

Fact $\quad \varphi_\varepsilon * f = f * \varphi_\varepsilon$

Feeling: $\varphi_\varepsilon * f$ "smooths out" f and approximates f as $\varepsilon \searrow 0$.

Why $\quad (\varphi_\varepsilon * f)(x) = \int_{-\infty}^{\infty} f(t) \, \varphi_\varepsilon(x-t) \, dt$

$$DQ = \frac{(\varphi_\varepsilon * f)(x+h) - (\varphi_\varepsilon * f)(x)}{h} = \int_{-\infty}^{\infty} f(t) \underbrace{\frac{\varphi_\varepsilon(x+h-t) - \varphi_\varepsilon(x-t)}{h}}_{\substack{\text{very nice!} \\ \text{dies away fast!}}} dt$$

$$\frac{d}{dx}(\varphi_\varepsilon * f)(x) = \frac{d}{dx}(f*\varphi_\varepsilon)(x) = (f*\varphi_\varepsilon')(x)$$
$$= (\varphi_\varepsilon' * f)(x)$$

Can diff through the convolution to see that

$\varphi_\varepsilon * f$ is C^∞-smooth even if f is merely continuous!

Works if f is continuous and bounded on $\mathbb{R}$ (or even if f piecewise continuous and of "exponential type," meaning that $|f(x)| \le Ke^{c|x|}$ on $\mathbb{R}$).

Thm: C^∞-smooth are "dense" among bounded continuous fcns on $\mathbb{R}$.

Why: $(\varphi_\varepsilon * f)(x) - f(x) \leftarrow$ multiply by $1 = \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\,dt$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(t) f(x-t) - \varphi_\varepsilon(t) f(x) \quad dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\varphi_\varepsilon(t)}_{\substack{\text{small} \\ |t|>\delta \\ \text{when } \varepsilon \text{ small enough}}} \Big[\underbrace{f(x-t) - f(x)}_{\substack{\text{small for} \\ t \text{ small.}}} \Big] dt \quad |t|<\delta$$

$|f(t)| \le M$ on $\mathbb{R}$

Given $\varepsilon>0$, $\exists\ \delta$ such that $|f(x-t)-f(x)| < \frac{\varepsilon}{2}$, $|t|<\delta$.

$$= \int_{-\delta}^{\delta} \varphi_\varepsilon(t)\left[f(x-t)-f(x)\right]dt + \int_{|t|\ge\delta} \varphi_\varepsilon(t)\left[f(x-t)-f(x)\right]dt$$

$$\underbrace{\qquad\qquad}_{I_1} \qquad \underbrace{\qquad\qquad}_{I_2}$$

$$|I_1| \le \int_{-\delta}^{\delta} \varphi_\varepsilon(t)\, \underbrace{|f(x-t)-f(x)|}_{<\frac{\varepsilon}{2}}\, dt < \frac{\varepsilon}{2} \underbrace{\int_{-\delta}^{\delta} \varphi_\varepsilon(t)\, dt}_{< \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\,dt = 1}$$

Get $|I_1| < \frac{\varepsilon}{2}$.

$$|I_2| \le \int_{|t|\ge\delta} \varphi_\varepsilon(t)\, \underbrace{|f(x-t)-f(x)|}_{\le 2M}\, dt$$

Claim $\displaystyle\int_{|t|\ge\delta} \varphi_\varepsilon(t)\,dt \to 0$ as $\varepsilon \to 0$

$$= 2\int_{\delta}^{\infty} \varphi_\varepsilon(t)\,dt = 2\int_{\delta}^{\infty} \frac{1}{\varepsilon} e^{-(t/\varepsilon)^2}\,dt$$

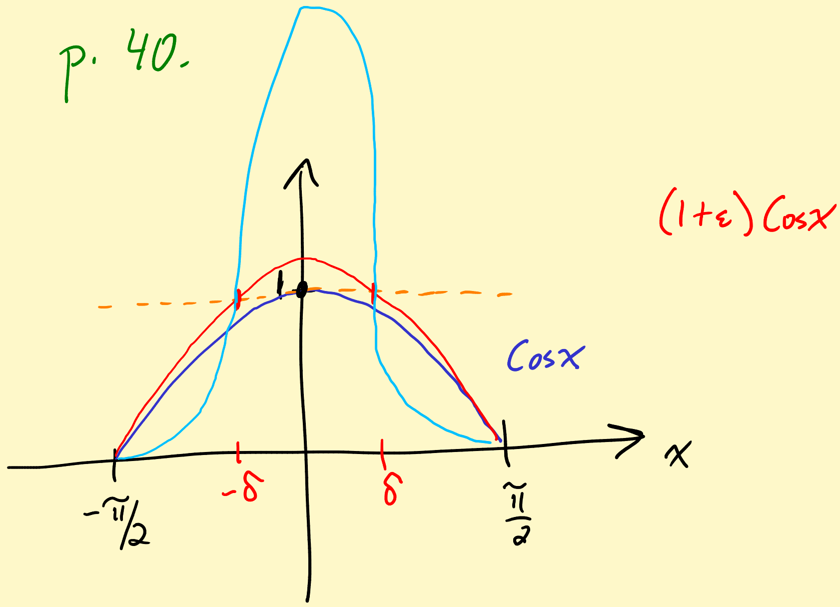
When $\varepsilon < \delta$ $\quad e^{-t^2/\varepsilon^2} < e^{-t/\varepsilon}$ when $t > \delta$.

Can calculate $\displaystyle\int_{\delta}^{\infty} e^{-t/\varepsilon}\,dt$

and see $\to 0$ as $\varepsilon \to 0$.

Can make $|I_2| < \frac{\varepsilon}{2}$ too. See $\underbrace{(\varphi_\varepsilon * f)}_{C^\infty\text{-smooth}}(x) \to f(x)$

Reading Stein, p. 40.



$\frac{1}{C}\left[(1+\varepsilon)\cos x\right]^n$ bump fcn. Divide by C to make area under graph 1.

$\hat{f}(n) = 0$ for all n. Then $f \perp$ Trig polys.

$f \perp$ Big bump.

$$0 = \int_{-\pi}^{\pi} f(t)\,(\text{Big bump at } x)\, dt$$

$\longrightarrow f(x)$