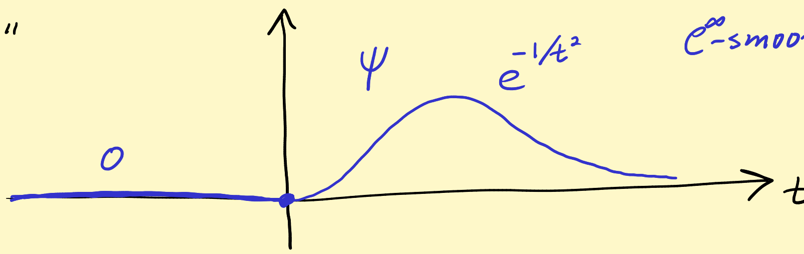
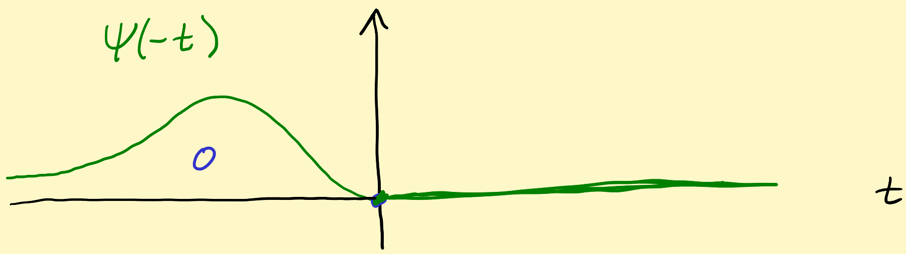
Lecture 20 Review: Parseval's identity

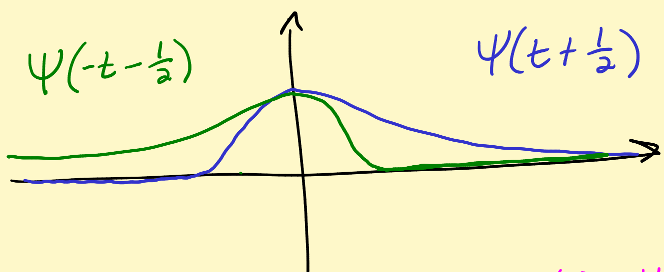
Fantastic "good kernel"



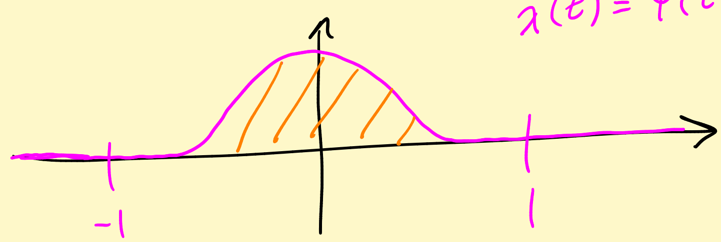
C^∞-smooth!



Trick



$$\lambda(t) = \psi(t+\tfrac{1}{2})\,\psi(-t-\tfrac{1}{2})$$

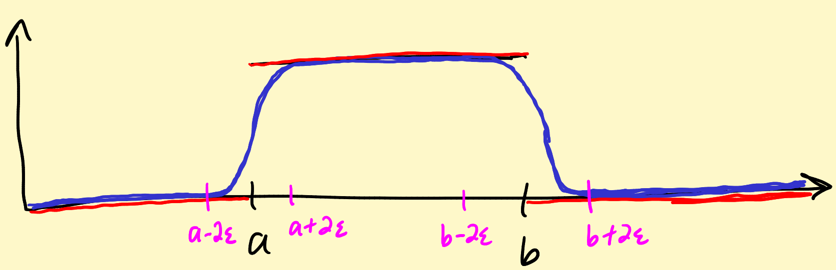


Area = C

Bump fcn $\varphi(t) = \frac{1}{C}\lambda(t)$ is $C_0^\infty(\mathbb{R})$. ← C^∞ with compact support

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t/\varepsilon)$$

History: $\varphi_\varepsilon \longrightarrow$ Dirac delta fcn δ as a "distribution"



C_0^∞ approximation to $\chi_{[a,b]}$

Easy: $\varphi_\varepsilon * \chi_{[a,b]}$

Parseval's for piecewise continuous fcns

Continuous 2π-periodic case

$$\int_{-\pi}^{\pi} \Big| f - \sum_{-N}^{N} \hat{f}(n) e^{inx} \Big|^2 dx \le \int_{-\pi}^{\pi} \Big| f - \sum_{-N}^{N} A_n e^{inx} \Big|^2 dx$$

Best L^2-approx to f among all trig polys of deg N.

Use Fejér's on this side. Can approx f uniformly to within any $\varepsilon > 0$ via a Fejér trig poly. (Might need a big N.)

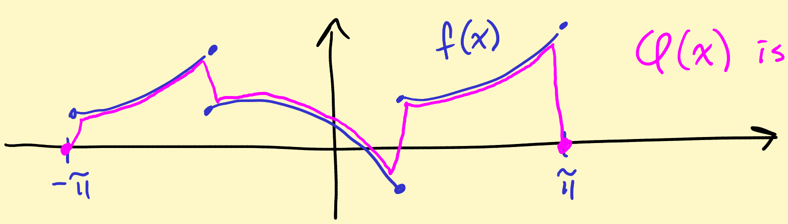
$$\text{LHS} = \underbrace{\int_{-\pi}^{\pi} |f|^2 dx - 2\pi \sum_{-N}^{N} |\hat{f}(n)|^2}_{\ge 0}$$

<u>Note</u> a Fejér poly of deg N is a also a trig poly of deg $m > N$ (by letting coeff A_k with $|k| > N$ be <u>zero</u>.)

Conclude: LHS $\to 0$ as $N \to \infty$.

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 dx$$

Trick for piecewise continuous f:



$\varphi(x)$ is continuous. Extend to be 2π-periodic

Given $\varepsilon > 0$.

Can make $\left(\int_{-\pi}^{\pi} |f-\varphi|^2\,dx\right)^{1/2} < \frac{\varepsilon}{2}$

<u>Fact</u> L^2 norm $\|f\| = \left(\int_{-\pi}^{\pi} |f|^2\,dx\right)^{1/2}$ satisfies

the Δ-ineq.

$$\left\| f - \sum_{-N}^{N} \hat{f}(n) e^{inx} \right\| \le \underbrace{\left\| f - \sum_{-N}^{N} A_n e^{inx} \right\|}_{\left\| f - \varphi + \varphi - \sum_{-N}^{N} A_n e^{inx} \right\|}$$

$$\le \underbrace{\| f - \varphi \|}_{< \frac{\varepsilon}{2}} + \underbrace{\left\| \varphi - \sum_{-N}^{N} A_n e^{inx} \right\|}_{\text{φ cont 2π-periodic. Can get Fejér trig polys to make this $< \frac{\varepsilon}{2}$}}$$

Get Parseval's for piecewise cont. f.

This same argument shows Fourier series $\longrightarrow f$ in the L^2-norm when f is piecewise continuous.

Riemann showed F.S $\to f$ in L^2 when f is Riemann integrable.

Lebesgue had to save the day because:

There exist sequences f_n of Riemann integrable fcns that are Cauchy seq in L^2-norm, that do not converge to a Riemann integrable fcn in L^2. But the limit does exists as a Lebesgue integrable fcn in L^2.

Fact A function f on $[a,b]$ is Riemann integrable if and only if

1) f is bounded on $[a,b]$

2) f is continuous almost everywhere on $[a,b]$.

Def^n "Almost everywhere" means at all points except perhaps on a set of measure zero.

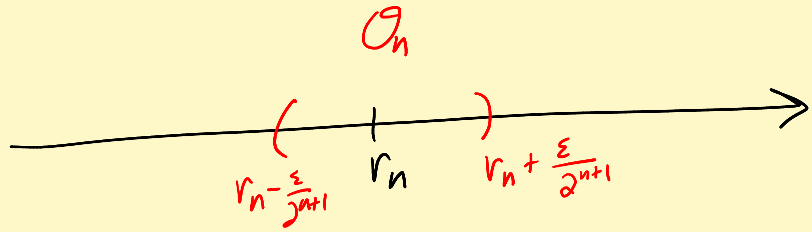
Def^n: $S' \subset [a,b]$ has measure zero, if for $\varepsilon > 0$, there are open intervals (α_n, β_n) such that

$$S' \subset \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (\beta_n - \alpha_n) < \varepsilon.$$

Classic example of a set of measure zero:

$\underbrace{\text{Rational \#'s}}_{\mathbb{Q}}$ in $\mathbb{R}$.

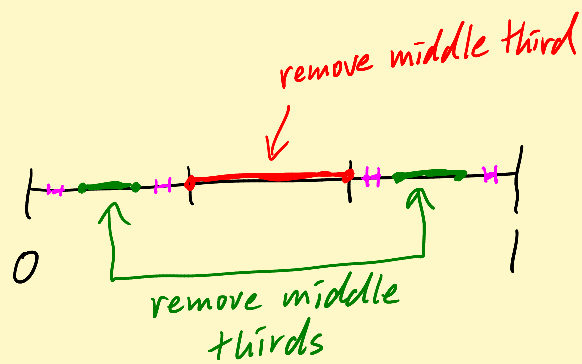
Easy: $\mathbb{Q} = \{ r_n : n \in \mathbb{N} \}$ is countable. $\varepsilon > 0$



$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} O_n$$ ← sum of interval lengths is $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

Pf of Fact about Riemann integrals: W. Rudin, Principles of Mathematical Analysis

<u>Cantor set</u>



repeat forever.

What's left is called the Cantor set.

<u>Facts</u> 1) It is <u>uncountable</u>

2) It is a set of <u>measure zero</u>

3) It is a <u>perfect</u> set: every pt in the set is a limit pt of the set.