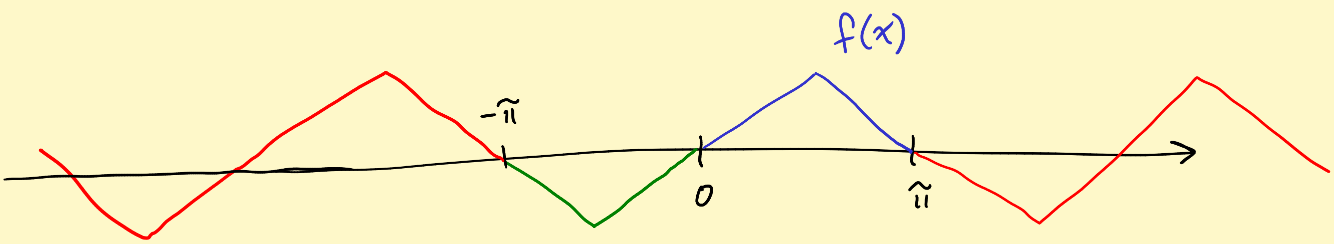
Lecture 21 D'Alembert's solution to the string problem

See p.8 (1.2) Stein

Jacob Bernoulli: $u(x,t) = \frac{1}{2}\left[f^*(\underbrace{x+ct}_{v}) + f^*(\underbrace{x-ct}_{w}) \right]$ books: ξ, η



f^* is the <u>odd</u> 2π-periodic extension of f from $[0,\pi]$.

d'Alembert: $\begin{aligned} v &= x+ct \\ w &= x-ct \end{aligned}$ $\boxed{u_{tt} = c^2 u_{xx}}$

Chain rule: $u_x = \frac{\partial u}{\partial v} \underbrace{\frac{\partial v}{\partial x}}_{=1} + \frac{\partial u}{\partial w} \underbrace{\frac{\partial w}{\partial x}}_{=1} = u_v + u_w$

$$u_{xx} = \left(\frac{\partial u_v}{\partial v} \underbrace{\frac{\partial v}{\partial x}}_{1} + \frac{\partial u_v}{\partial w} \underbrace{\frac{\partial w}{\partial x}}_{1} \right) + \left(\frac{\partial u_w}{\partial v} \underbrace{\frac{\partial v}{\partial x}}_{1} + \frac{\partial u_w}{\partial w} \underbrace{\frac{\partial w}{\partial x}}_{1} \right)$$

$$= u_{vv} + \underbrace{u_{wv} + u_{vw}}_{= \text{ if } C^2\text{-smooth. Assume this.}} + u_{ww}$$

$$\boxed{u_{xx} = u_{vv} + 2u_{wv} + u_{ww}}$$

$\begin{cases} v = x+ct \\ w = x-ct \end{cases}$

Repeat for t partials. $u_t = \frac{\partial u}{\partial v} \underbrace{\frac{\partial v}{\partial t}}_{c} + \frac{\partial u}{\partial w} \underbrace{\frac{\partial w}{\partial t}}_{-c}$

$\vdots$

$$\boxed{u_{tt} = c^2 \left(u_{vv} - 2u_{wv} + u_{ww} \right)}$$

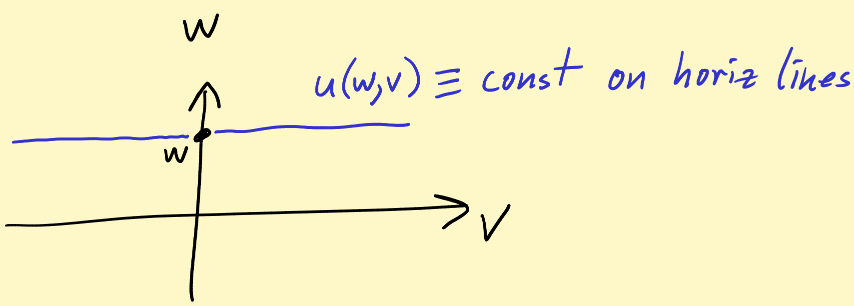
Plug into wave eqn. Cancel easy stuff.

Wow! $\quad -2c^2 u_{wv} = 2c^2 u_{wv}$

$$\boxed{u_{wv} = 0} \leftarrow \text{easy!}$$

Freshman calculus fact: $\frac{d}{dx} h(x) \equiv 0 \Rightarrow h \equiv \text{const.}$ (MVT)

$$\boxed{\frac{\partial u}{\partial v} \equiv 0}$$



Hmmm. Different w's might lead to different constants.

$u(w,v) = C(w) \leftarrow$ a fcn of w.

And conversely, if $u(w,v) = C(w)$, then $\frac{\partial u}{\partial v} \equiv 0$.

$$\boxed{\text{Sol}^n \text{ to } \frac{\partial u}{\partial v} \equiv 0 \text{ is } u = \text{Arb fcn of } w.}$$

Now $\quad u_{wv} = 0$

$$\boxed{\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial v}\right] = 0}$$

Conclude $\frac{\partial u}{\partial v} = C(v) \leftarrow \frac{\partial}{\partial w} \equiv 0 \implies = \text{fcn of other var } v$

Step 1 Let $p(v)$ be an antiderivative of $C(v)$.

So $p'(v) = C(v)$.

Then $\frac{\partial u}{\partial v} - C(v) = 0$

$$\frac{\partial}{\partial v}\Big[\underbrace{u - p(v)}_{= \text{ fcn of } w,\ K(w)}\Big] = 0$$

Aha! $u - p(v) = K(w)$ $\quad u(w,v) = p(v) + K(w)$

D'Alembert's sol^n

$$\boxed{u(x,t) = \varphi(x+ct) + \psi(x-ct)}$$

$$\frac{\partial u}{\partial t} = \varphi'(x+ct)c + \psi'(x-ct)(-c)$$

I.C. $\begin{cases} u(x,0) = \varphi(x) + \psi(x) \overset{\text{want}}{=} f(x) \\ \frac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) = g(x) \end{cases}$

Harpsichord. $g(x) \equiv 0$. $\begin{cases} \varphi(x) + \psi(x) = f(x) & (A) \\ \varphi'(x) - \psi'(x) = 0 & (B) \end{cases}$

$$(A) \qquad f(x) = \varphi(x) + \psi(x)$$

$$\int_0^x (B): \qquad 0 = \varphi(x) - \psi(x)$$

$$+ \qquad f(x) = 2\varphi(x). \qquad \text{So} \qquad \boxed{\varphi(x) = \tfrac{1}{2} f(x)}$$

$$- \qquad f(x) = 2\psi(x). \qquad \text{So} \qquad \boxed{\psi(x) = \tfrac{1}{2} f(x).}$$

B.C. $\quad u(0,t) = \varphi(ct) + \psi(-ct) = 0 \quad$ for all t

Hmmm. $\quad \psi(-ct) = -\varphi(ct)$

$$\tfrac{1}{2} f(-ct) = -\tfrac{1}{2} f(ct)$$

need odd extension!

$$u(\pi, t) = \varphi(\pi + ct) + \psi(\pi - ct)$$

Conclude we also need the 2π-periodic extension.

Now everything works!

$$u(x,t) = \tfrac{1}{2}\left[f^*(x+ct) + f^*(x-ct) \right]$$

See Stein p. 8-10 for answer when $g \not\equiv 0$. (Piano case.)

<u>Frosting on cake</u> Isoperimetric inequality and Weyl's equidistribution theorem

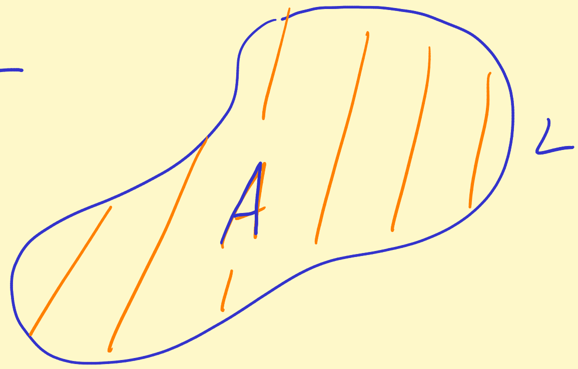
Circle

Perimeter $= 2\pi r = L \quad \left(r = \frac{L}{2\pi}\right)$

Area $= \pi r^2$

$$= \pi \left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$

Fence of length L

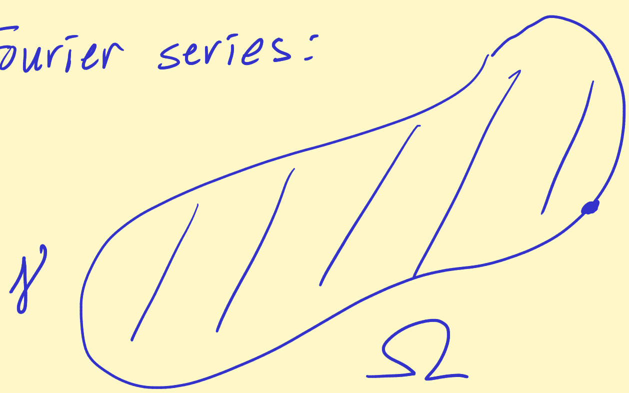


$$\boxed{A \leq \frac{L^2}{4\pi}}$$

Isoperimetric ineq

<u>and</u>, if $A = \frac{L^2}{4\pi}$, it <u>must</u> be a circle.

Connection to Fourier series:



$\mathcal{C}^2$-smooth boundary curve of length 2π.

γ: $(x(t), y(t))$ parametrize with respect to arc length.
$0 \leq t \leq 2\pi$

$x(t), y(t)$ 2π- periodic, $\mathcal{C}^2$-smooth.

Area = ?   Green's theorem!

$$\int_\gamma \underset{\substack{\uparrow \\ -\frac{y}{2}}}{P}\,dx + \underset{\substack{\uparrow \\ Q=\frac{x}{2}}}{Q}\,dy = \iint_\Omega \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{want } = 1} dx\,dy$$

$$\underbrace{\frac{1}{2}\int_\gamma -y\,dx + x\,dy}_{\substack{\text{inner product} \\ \text{on } [0,2\pi]!}} = \iint_\Omega \underbrace{\tfrac{1}{2}+\tfrac{1}{2}}_{1}\,dx\,dy = \text{Area}(\Omega)$$