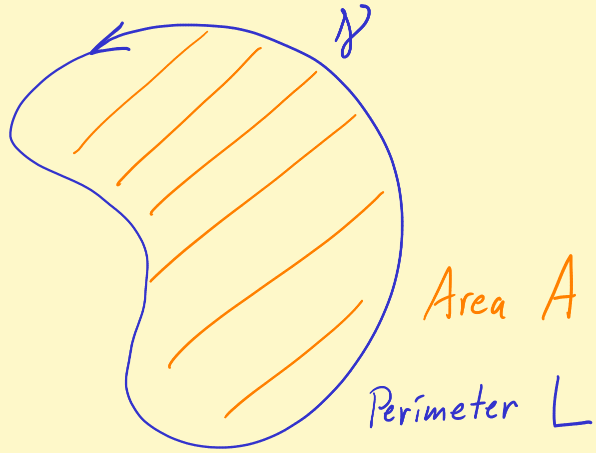
Lecture 22 The isoperimetric inequality via Fourier series!



γ simple closed curve, length L

$$\boxed{A \le \frac{L^2}{4\pi}}$$

← equality only when circle

Start reading Chap 4

Circle case: $A = \pi r^2$

$L = 2\pi r \leftarrow r = \frac{L}{2\pi}$

$$A = \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$

Key ingredients 1) Green's theorem

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{2} dx\,dy$$

$Q = x$ $P = -y$

$$\int_\gamma -y\,dx + x\,dy = 2(\text{Area of } \Omega)$$

2)

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$F(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

Parseval's $\Longrightarrow$

$$\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)F(x)\,dx}_{L^2[-\pi,\pi] \text{ inner product}} = \underbrace{2a_0A_0 + \sum_{n=1}^{\infty}(a_nA_n + b_nB_n)}_{\ell^2 \text{ inner product}}$$

ℓ^2 is the Hilbert space of square summable series.

Pf: Parseval's for $f+F$

$$\begin{array}{r} f+F \\ -\quad f-F \\ \hline \text{New formula} \end{array}$$

Complex version:
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{F(x)}\,dx = \sum_{n=-\infty}^{\infty} c_n \overline{C_n}$$

Steps to simplify calculations

1) Rescale axes so that $L = 2\pi$.

   Isop. ineq becomes $\boxed{A \le \pi}$

2) Parametrize γ: $z(t) = (x(t), y(t))$

   via arc length t, $-\pi \le t \le \pi$.

3) Assume γ C^2-smooth (and 2π-periodic and C^2-smooth on $\mathbb{R}$)

Notation $x(t), y(t)$

$$\hat{x}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-int}\,dt, \quad \hat{y}(n).$$

Key ingredient: $\hat{\dot{x}}(n) = in\,\hat{x}(n)$

how to remember

$$x(t) = \sum_{-\infty}^{\infty} \hat{x}(n)\, e^{int}$$

$$\dot{x}(t) = \sum_{-\infty}^{\infty} in\,\hat{x}(n)\, e^{int}$$

How to prove: complex version of integration by parts

$$\hat{x}(n) = \int_{-\pi}^{\pi} \underbrace{e^{-int}}_{u} \underbrace{\dot{x}(t)\, dt}_{dv} \qquad du = -ine^{-int}\, dt \quad v = x(t)$$

Important observation: Since t is arc length, $\frac{ds}{dt} = 1$.

So $$ds = \underbrace{\sqrt{(\dot{x})^2 + (\dot{y})^2}}_{1}\, dt$$

So $$\dot{x}(t)^2 + \dot{y}(t)^2 \equiv 1 \quad \underline{\text{and}}$$

$$\int_{-\pi}^{\pi} \dot{x}(t)^2 + \dot{y}(t)^2 \, dt = 2\pi$$

$$\overset{\text{Parsevals}}{=} 2\pi \sum_{-\infty}^{\infty} \left(|in\,\hat{x}(n)|^2 + |in\,\hat{y}(n)|^2 \right)$$

Get
$$\boxed{\pi = \pi \sum_{-\infty}^{\infty} n^2 \left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2 \right)}$$

Recall $$A = \frac{1}{2} \int_{\gamma} -y \, dx + x\, dy$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} -y(t)\,\dot{x}(t) + x(t)\,\dot{y}(t) \quad dt$$

Parseval's improved identity

$$= \frac{1}{2}\left[2\pi \sum_{-\infty}^{\infty} -\hat{y}(n)\overline{\left(in\hat{x}(n)\right)} + \hat{x}(n)\overline{\left(in\hat{y}(n)\right)}\right]$$

Want to see $A \leq \pi$

$$\pi - A = \pi \sum_{-\infty}^{\infty} n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right)$$
$$- \pi \sum_{-\infty}^{\infty} in\left(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)}\right)$$

Mind blowing thing : $\pi - A = \underbrace{\text{Sum of squares!}}_{\text{all zero} \Rightarrow \text{circle!}} \geq 0$

Write
$$\hat{x}(n) = \alpha_n + i\beta_n$$
$$\hat{y}(n) = a_n + ib_n$$

Algebra
$$\pi - A = \pi \sum_{-\infty}^{\infty}\left[n^2\left(\alpha_n^2 + \beta_n^2 + a_n^2 + b_n^2\right) - 2n\left(\beta_n a_n - \alpha_n b_n\right)\right]$$

$$= \pi \sum_{-\infty}^{\infty}\left[(n\alpha_n + b_n)^2 + (n\beta_n - a_n)^2 + (n^2-1)(a_n^2 + b_n^2)\right]$$

Convention: Choose origin so that
$$\left.\begin{aligned}\hat{x}(0) &= \text{ave of } x(t) = 0\\ \hat{y}(0) &= \text{ave of } y(t) = 0\end{aligned}\right\}$$

kills the one negative term

So $n^2-1 \geq 0$, $n = \pm 1, \pm 2, \ldots$

Voila! $\pi - A \geq 0$. ✓ Isop. ineq ✓

what if $A = \pi$? Last term shows $a_n = 0$, $b_n = 0$ $n = \pm 2, \pm 3, \ldots$

(Maybe $a_{\pm 1}$, $b_{\pm 1}$ not zero)

Next, first two terms now show that $\alpha_n = 0$, $\beta_n = 0$ when $n = \pm 2, \pm 3, \ldots$

Aha!

$$\left.\begin{aligned} x(t) &= c_0 + c_{-1} e^{-it} + c_1 e^{it} \\ y(t) &= c_0 + c_{-1} e^{-it} + c_1 e^{it} \end{aligned}\right\} \text{Show this traces out a circle}$$

Little facts that help: Because $x(t)$ and $y(t)$ real,

$$\hat{x}(-n) = \overline{\hat{x}(n)} \quad \leftarrow \quad \alpha_{-1} + i\beta_{-1} = \alpha_1 - i\beta_1$$

$$\hat{y}(-n) = \overline{\hat{y}(n)} \quad \leftarrow \quad a_{-1} + i b_{-1} = a_1 - i b_1$$

Why: $$\hat{x}(-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t)\, \underbrace{e^{-i(-n)t}}_{} \, dt$$

$$e^{int} = \overline{e^{-int}}$$

Play around. Get

$$\begin{cases} x(t) = \alpha_1 \cos t - \beta_1 \sin t \\ y(t) = \beta_1 \cos t + \alpha_1 \sin t \end{cases} \quad \text{Circle!}$$

$n=1$ term : $(1\cdot\alpha_1 + b_1)^2 + (1\cdot\beta_1 - a_1)^2 + \underbrace{(1^2-1)}_{0}(a_1^2+b_1^2)$

$\alpha_1 = -b_1 \qquad \beta_1 = a_1$

$n=-1$ term ; $(-1\,\alpha_{-1} + b_{-1})^2 + (-1\cdot\beta_{-1} - a_{-1})^2 + \underbrace{(1^2-1)}_{0}(a_{-1}^2 + b_{-1}^2)$

$b_{-1} = \alpha_{-1} \qquad a_{-1} = -\beta_{-1}$