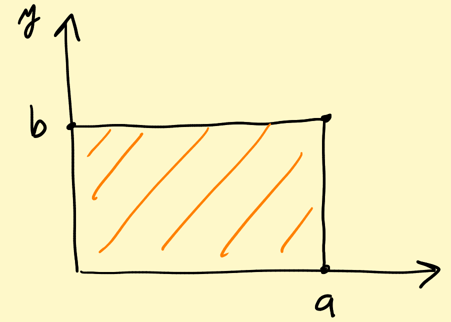
Lecture 23 Vibrating square drum



Tightly stretched membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$u(x,y,t) = 0$ on edges (BC)

$$\begin{cases} u(x,y,0) = f(x,y) \\ \frac{\partial u}{\partial t}(x,y,0) = g(x,y) \end{cases}$$ known (IC)

Try $u(x,y,t) = X(x)Y(y)T(t)$. Take $c=1$.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Divide by XYT

$$XYT'' = X''YT + XY''T$$

$$\lambda = \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Let $t = t_0$. See RHS = const. So LHS = same constant λ

T-problem: $T'' - \lambda T = 0$

X, Y: $\underbrace{\frac{X''}{X}}_{\text{fcn of } x} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{fcn of } y} = \mu$

a constant

X-problem: $X'' - \mu X = 0$ BC $\begin{cases} X(0) = 0 \\ X(a) = 0 \end{cases}$

Y- problem : $Y'' - (\lambda - \mu) Y = 0$ BC $\begin{cases} Y(0) = 0 \\ Y(b) = 0 \end{cases}$

BC : $u(a, y, t) = X(a) Y(y) T(t) = 0$ on right side edge

don't want these $\equiv 0$

need $X(a) = 0$

X - prob : To get non-zero sol^{n}s, need $\mu_n = -\left(\frac{n\pi}{a}\right)^2$

$$X_n(x) = \sin \frac{n\pi x}{a}$$

Y - prob : $Y'' - (\lambda - \mu_n) Y = 0$ $Y(0) = 0,\ Y(b) = 0.$

Only get non-zero sol^{n}s when

$$\lambda - \mu_n = -\left(\frac{m\pi}{b}\right)^2$$

Get $Y_m(y) = \sin \frac{m\pi y}{b}$.

T - prob : $T'' - \lambda T = 0$ $\lambda = -\left(\frac{m\pi}{b}\right)^2 + \mu_n$

$$\lambda_{nm} = -\left(\frac{m\pi}{b}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

Let $\lambda_{nm} = -\omega_{nm}^2$

ODE : $r^2 + \omega_{nm}^2 = 0$ $r = \pm i\, \omega_{nm}$

$$T_{nm}(t) = A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t$$

where $\omega_{nm} = \sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2}$

Basic sol^n

$$u_{nm}(x,y,t) = X_n(x)\, Y_m(t)\, T_{nm}(t)$$

$$= \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\left[A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t\right]$$

Sound frequency of this motion is ω_{nm}!

Basic mode of oscillation.

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{nm}(x,y,t)$$

IC: $$u(x,y,0) = \sum_{n,m=1}^{\infty} A_{nm}\underbrace{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}}_{\text{orthogonal}} \overset{\text{want}}{=} f(x,y)$$

2-Dim Fourier sine series

$\iint_{\text{Rectangle}} FG\, dA$

$$\frac{\partial u}{\partial t}(x,y,0) = \sum_{n,m=1}^{\infty} \omega_{nm} B_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \overset{\text{want}}{=} g(x,y)$$

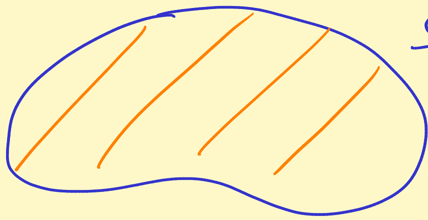
Facts Overtones of rec drums are discordant with the fundamental pitch of drum

$$\omega_{11} = \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}. \quad \text{Noise!}$$

Question: Which rectangular drum with given area A has the lowest pitch?

Ans: The square one! $a=b$. $a^2=A$.

Famous problem Can you hear the shape of drum?



Ω

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

"Eigenfunctions of the Laplacian" $\Delta\varphi = \lambda\varphi$, $\varphi|_{\partial\Omega}=0$

λ's determine the pitches of the overtones.

Do they determine Ω?

Carolyn Gordon : No!

Circular drum problem: λ's related to zeroes of Bessel fcns!

Ques Can you tell if you have circular drum?