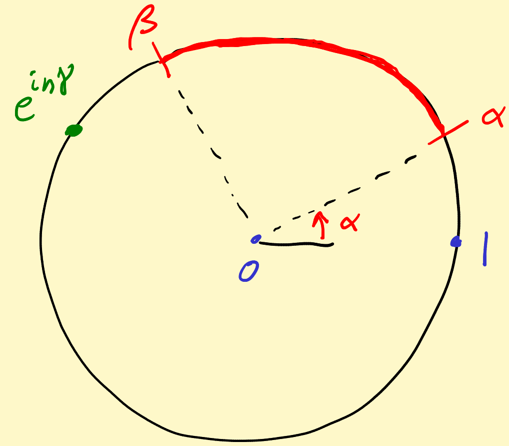
Lecture 24 Weyl's equidistribution theorem



Herman Weyl "Vile"
André Weil "Vay"

Assume γ is an irrational multiple of π.
Then the points $e^{in\gamma}$ $(n \in \mathbb{N})$ are equidistributed on the unit circle $C_1(0)$.

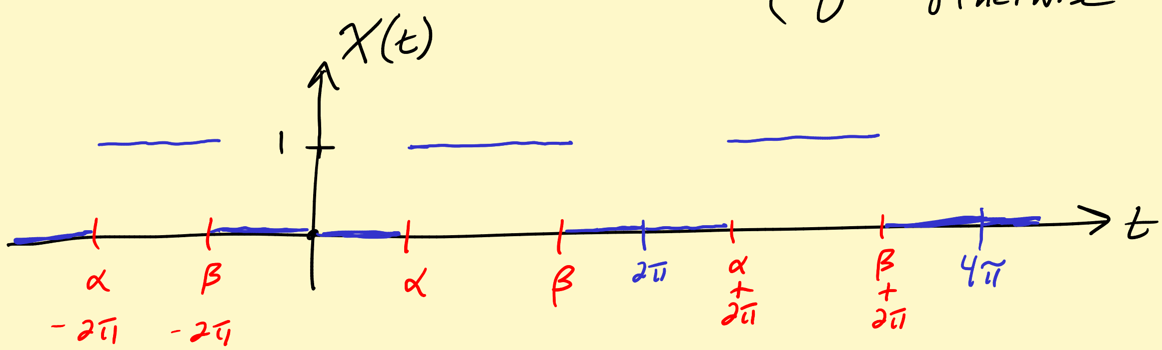
Define $$R_N = \frac{\#\text{times } e^{in\gamma} \text{ falls in } (\alpha,\beta) \text{ as } n=1,2,3,\dots,N}{N}$$

$$\xrightarrow[\text{Thm}]{} \frac{\beta-\alpha}{2\pi} \quad \text{as } N \to \infty.$$

Pf Define $$\chi(t) = \chi(e^{it}) = \chi_{(\alpha,\beta)} = \begin{cases} 1 & e^{it} \in \{e^{i\theta} : \alpha < \theta < \beta\} \\ 0 & \text{otherwise} \end{cases}$$

abuse of notation!

$$= \begin{cases} 1 & t \in (\alpha,\beta) \text{ modulo } 2\pi \\ 0 & \text{otherwise} \end{cases}$$



Notice that $$R_N = \frac{\sum_{n=1}^N \chi(e^{in\gamma})}{N} = \frac{\sum_{n=1}^N \chi(n\gamma)}{N}$$

hmmm $$= \frac{1}{2\pi} \cdot \underbrace{\left(\frac{2\pi}{N}\right)}_{\Delta t} \sum_{1}^{N} \chi(n\gamma)$$ looks like a Riemann sum

$$\xrightarrow[\text{hope}]{} \frac{1}{2\pi}\int_0^{2\pi} \chi(t)\,dt = \frac{\beta-\alpha}{2\pi} \quad \checkmark$$

Weyl's lemma If $f(t)$ is continuous and 2π-periodic, then

$$\frac{1}{N}\sum_{n=1}^{N} \underbrace{f(e^{in\gamma})}_{f(n\gamma)} \longrightarrow \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt.$$

Pf: Step 1 True for trig polys

$$A_0 + \sum_{n=1}^{N}(A_n \cos nt + B_n \sin nt) = \underbrace{\sum_{n=-N}^{N} C_N e^{int}}_{\text{use } \mathbb{C} \text{ version}}$$

(Note: C_N don't have to be the Fourier coeff.)

Step 0 True for $f(t) \equiv 1$:

$$R_N = \frac{\overbrace{1+1+1+\cdots+1}^{N\text{-times}}}{N} = 1 \stackrel{?}{=} \frac{1}{2\pi}\int_0^{2\pi} 1\,dt. \quad \text{Yes!}$$

Step .5 True for the other basis fcns for trig polys.

Write $f_m(t) = e^{imt}$. Then $f_m(e^{in\gamma}) = e^{imn\gamma} = \underline{(e^{im\gamma})^n}$

$$R_N = \frac{f_m(e^{i\gamma}) + f_m(e^{i2\gamma}) + \cdots + f_m(e^{iN\gamma})}{N}$$

$$= \frac{(e^{im\gamma})^1 + (e^{im\gamma})^2 + \cdots + (e^{im\gamma})^N}{N}$$ geom series!

$$= (e^{im\gamma}) \left[\frac{1 + (e^{im\gamma}) + (e^{im\gamma})^2 + \cdots + (e^{im\gamma})^{N-1}}{N} \right]$$

$$= \frac{(e^{im\gamma})}{N} \cdot \frac{1 - (e^{im\gamma})^{N-1+1}}{1 - e^{im\gamma}}$$

only place we need $\gamma =$ irrat mult π to see denom $\neq 0$

Important: $e^{i\theta} = 1 \iff \theta = 2\pi n$ for some $n \in \mathbb{Z}$.

Now $$|R_N| = \frac{\overbrace{|e^{im\gamma}|}^{=1}}{N} \frac{|1 - e^{im\gamma N}|}{|1 - e^{im\gamma}|}$$ $\leq 1 + 1 = 2$

$$\leq \frac{2}{N} \frac{1}{\underbrace{|1 - e^{im\gamma}|}_{\text{not zero}}} \to 0$$ as $N \to \infty$.

Big question: $$\frac{1}{2\pi} \int_0^{2\pi} e^{imt} \, dt \overset{?}{=} 0$$ Yes!

Step 0.75 True for $1, e^{imt}$. Hence true for linear combos of these, i.e., trig polys!

Step 2 Use Fejér's to get Weyl's lemma. (f cont 2π-periodic)

Let $\varepsilon > 0$, Fejér's $\Rightarrow \exists$ trig poly $P(t)$ with

$$|f(t) - P(t)| < \frac{\varepsilon}{3} \quad \text{for all } t$$

$$\mathcal{E}_N = \left| \frac{1}{N} \sum_{n=1}^{N} f(e^{in\gamma}) - \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt \right| \quad \xleftarrow{} \text{want} \to 0 \text{ as } N \to \infty$$

$$= \Bigg| \underbrace{\frac{1}{N}\sum_{n=1}^{N}\left(f(e^{in\gamma}) - P(e^{in\gamma})\right)}_{T_1} + \underbrace{\left(\frac{1}{N}\sum_{n=1}^{N} P(e^{in\gamma}) - \frac{1}{2\pi}\int_0^{2\pi} P(t)\,dt\right)}_{T_2}$$

$$+ \underbrace{\left(\frac{1}{2\pi}\int \left(P(t) - f(t)\right) dt\right)}_{T_3} \Bigg|$$

$$|\mathcal{E}_N| \le |T_1| + |T_2| + |T_3|$$

$$|T_1| \le \frac{\overbrace{\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \cdots \frac{\varepsilon}{3}}^{N\text{-times}}}{N} = \frac{\varepsilon}{3} \quad \checkmark$$

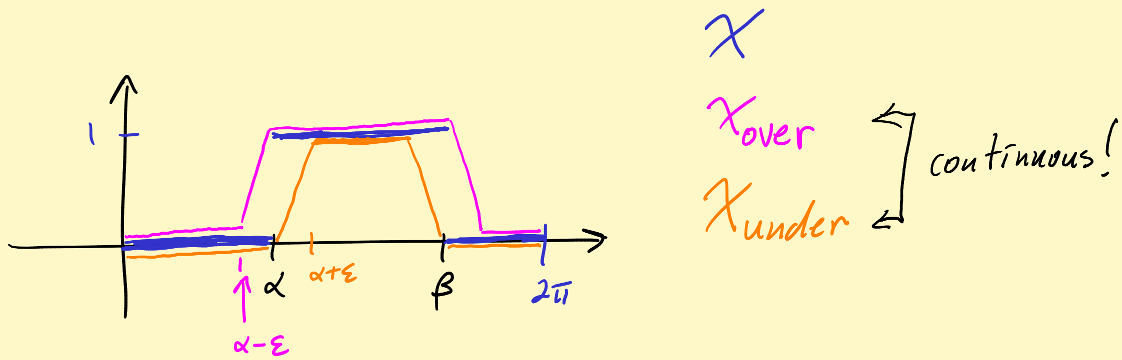
$|T_2| \to 0$ as $N \to \infty$ because we know Weyl's lemma for trig polys. So $\exists N_0$ such that $|T_2| < \frac{\varepsilon}{3}$ when $\underline{\underline{N > N_0}}$.

$$|T_3| = \left| \frac{1}{2\pi}\int_0^{2\pi} P(t) - f(t)\,dt \right| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|P(t) - f(t)|}_{< \frac{\varepsilon}{3}}\,dt$$

$$\le \frac{1}{2\pi} \cdot \frac{\varepsilon}{3} \cdot 2\pi = \frac{\varepsilon}{3} \quad \checkmark$$

So $|\mathcal{E}_N| < \varepsilon$ when $N > N_0$.

Last step



$$\chi_u \le \chi \le \chi_o$$

$$\frac{1}{N}\sum_{n=1}^{N}\chi_u(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N}\chi(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N}\chi_o(e^{in\gamma})$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\chi_u(t)\,dt}_{\frac{\beta-\alpha-\varepsilon}{2\pi}} \qquad\qquad \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\chi_o(t)\,dt}_{\frac{\beta-\alpha+\varepsilon}{2\pi}}$$

Hence $\displaystyle \frac{1}{N}\sum_{n=1}^{N}\chi(e^{in\gamma}) \xrightarrow[N\to\infty]{} \frac{\beta-\alpha}{2\pi}$ ✓

Pseudo-random numbers. How to make a computer throw darts $[0,1]$. Get 2 gigantic primes p, q. Take N from machine clock.

Throw dart: $N \cdot \frac{p}{q} \mod \mathbb{Z}$

Weyl's lemma for step fcns $\Longrightarrow$ Weyl's lemma for Riemann integrable fcns too because Riemann integrable fcns can be well approximated by linear combos of step fcns.